A. REVIEW OF PROBABILITY

A1. BASIC PROBABILITY

Cumulative Distribution Function: $F(x) = \Pr(X \leq x)$

Survival Function: $S(x) = 1 - F(x) = \Pr(X > x)$

Probability Density Function: $f(x) = \frac{d}{dx} F(x) = -\frac{d}{dx} S(x)$

Hazard Rate Function: $\lambda(x) = \frac{f(x)}{S(x)} = -\frac{d}{dx} \log S(x) \quad \rightarrow \quad S(x) = e^{-\int_{-\infty}^{x} \lambda(t) dt}$

Cumulative Hazard Function: $\Lambda(x) = \int_{-\infty}^{x} \lambda(t)\, dt \quad \rightarrow \quad S(x) = e^{-\Lambda(x)}$

A2. EXPECTATION & VARIANCE

Expectation (Discrete X): $E[X] = \sum x \Pr(X = x) \quad \rightarrow \quad E[g(X)] = \sum g(x) \Pr(X = x)$

Expectation (Continuous X): $E[X] = \int_{-\infty}^{\infty} x f(x)\, dx \quad \rightarrow \quad E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

Variance: $Var(X) = E\left[X^2\right] - E[X]^2$

Standard Deviation: $SD(X) = \sqrt{Var(X)}$

Covariance: $Cov(X, Y) = E[XY] - E[X]E[Y]$

Correlation Coefficient: $Corr(X, Y) = \dfrac{Cov(X, Y)}{SD(X)\ SD(Y)}$

Linear Combination: $W = aX + bY$

$E[W] = aE[X] + bE[Y]$

$Var(W) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X, Y)$

Sum of iid X: $S = X_1 + \cdots + X_n$

$E[S] = E[X_1 + \cdots + X_n] = nE[X]$

$Var(S) = Var(X_1 + \cdots + X_n) = nVar(X)$

A3. CONDITIONAL PROBABILITY

Conditional expectation (Discrete): $E[X \mid Y=y] = \sum x \Pr(X=x \mid Y=y)$

Conditional expectation (Continuous): $E[X \mid Y=y] = \int_{-\infty}^{\infty} x f(x \mid y) dx$

Law of total expectation: $E[X] = E[E[X|Y]]$

Law of Total Variance: $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

A4. COMMON DISTRIBUTIONS

Uniform distribution: $f_X(x) = \frac{1}{b-a}$ for $a \le x \le b$ $\rightarrow$ $E[X] = \frac{a+b}{2}$ $\quad Var(X) = \frac{(b-a)^2}{12}$

Exponential distribution: $f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$ for $x > 0$ $\rightarrow$ $E[X] = \theta$ $\quad Var(X) = \theta^2$

Normal approximation: $S \dot\sim \left(\mu = E[S],\ \sigma^2 = Var(S)\right)$ $\rightarrow$ $\Pr(S \le k) \approx N\left(\frac{k-\mu}{\sigma}\right)$

B. REVIEW OF FINANCIAL MATHEMATICS

B1. INTEREST RATES

Annual effective interest rate: i

Rate of discount: $d = \frac{i}{1+i} = iv = 1 - v$

Discounting rate: $v = \frac{1}{1+i} = 1 - d$

Continuously compounded interest rate: $\delta = \log(1+i)$

Nominal interest rate: $i^{(m)} = m\left((1+i)^{\frac{1}{m}} - 1\right)$

Nominal rate of discount: $d^{(m)} = m\left(1 - (1-d)^{\frac{1}{m}}\right)$

B2. PRESENT VALUES

PV of n-year certain annuity-due: $\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$ (1 at the beginning of each year)

PV of n-year certain annuity-immediate: $a_{\overline{n}|} = \frac{1-v^n}{i}$ (1 at the end of each year)

PV of n-year continuous certain annuity: $\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$ (1 per year continuously)

PV of n-year 1/m-thly certain annuity-due: $\ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{d^{(m)}}$ (1/m at the beginning of each of the nm periods)

C. SURVIVAL MODELS

C1. SURVIVAL FUNCTION

Complete future lifetime: T_x $\rightarrow$ The pdf of T_x is $f_x(t) = {}_tp_x\ \mu_{x+t}$.

Curtate future lifetime: $K_x = \lfloor T_x \rfloor$ $\rightarrow$ The pmf of K_x is $\Pr(K_x = k) = {}_kp_x\ q_{x+k}$.

Survival function: $S_x(t) = \Pr(T_x > t)$

Formulas:

$S_x(u+t) = S_x(u)\ S_{x+u}(t)$ $\rightarrow$ $S_{x+u}(t) = \dfrac{S_x(u+t)}{S_x(u)}$

$S_0(x+t) = S_0(x)\ S_x(t)$ $\rightarrow$ $S_x(t) = \dfrac{S_0(x+t)}{S_0(x)}$

Three conditions:

(1) $S_x(0) = 1$

(2) $\lim\limits_{t \to \infty} S_x(t) = 0$

(3) $S_x(t)$ must be a non-increasing function of t.

C2. ACTUARIAL NOTATION

Survival probability: ${}_tp_x = \Pr(T_x > t)$

Mortality probability: ${}_tq_x = 1 - {}_tp_x = \Pr(T_x \le t)$

Formulas:

$${}_{t+u}p_x = {}_tp_x\ {}_up_{x+t}$$

$${}_{t+u}q_x = {}_tq_x + {}_tp_x\ {}_uq_{x+t}$$

$${}_{t|u}q_x = {}_tp_x\ {}_uq_{x+t} = {}_tp_x - {}_{t+u}p_x = {}_{t+u}q_x - {}_tq_x$$

C3. LIFE TABLES

Number of lives: l_x

Number of deaths: ${}_td_x = l_x - l_{x+t}$

Formulas:

$${}_tp_x = \frac{l_{x+t}}{l_x}$$

$${}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x}$$

$${}_{t|u}q_x = \frac{{}_ud_{x+t}}{l_x} = \frac{l_{x+t} - l_{x+t+u}}{l_x}$$

C4. FORCE OF MORTALITY

Definition: $\mu_x(t) = \mu_{x+t} = \frac{f_x(t)}{S_x(t)} = -\frac{d}{dt}\log S_x(t)$

Condition: $\lim_{t\to\infty}\int_0^t \mu_s ds = \infty$

Formulas: $_tp_x = e^{-\int_0^t \mu_{x+s}ds} = e^{-\int_x^{x+t}\mu_s ds}$

$_tq_x = \int_0^t {}_sp_x\ \mu_{x+s}ds$

$_{t|u}q_x = \int_t^{t+u} {}_sp_x\ \mu_{x+s}ds$

C5. EXPECTED FUTURE LIFETIME

Expectations: $E[T_x] = \overset{o}{e}_x = \int_0^\infty t\ {}_tp_x\ \mu_{x+t}dt = \int_0^\infty {}_tp_x dt$

$E[K_x] = e_x = \sum_{k=1}^{\infty} k\ {}_kp_x\ q_{x+k} = \sum_{k=1}^{\infty} {}_kp_x$

$E[\min(T_x,n)] = \overset{o}{e}_{x:\overline{n}|} = \int_0^n t\ {}_tp_x\ \mu_{x+t}dt + n\,{}_np_x = \int_0^n {}_tp_x dt$

$E[\min(K_x,n)] = e_{x:\overline{n}|} = \sum_{k=1}^{n-1} k\ {}_kp_x\ q_{x+k} + n\,{}_np_x = \sum_{k=1}^{n} {}_kp_x$

Second moments: $E[T_x^2] = \int_0^\infty t^2\,{}_tp_x\ \mu_{x+t}dt = \int_0^\infty 2t\,{}_tp_x dt$

$E[K_x^2] = \sum_{k=1}^{\infty} k^2\,{}_kp_x\ q_{x+k} = \sum_{k=1}^{\infty}(2k-1)\,{}_kp_x = 2\sum_{k=1}^{\infty} k\ {}_kp_x - e_x$

$E\left[\min(T_x,n)^2\right] = \int_0^n t^2\,{}_tp_x\ \mu_{x+t}dt + n^2\,{}_np_x = \int_0^n 2t\,{}_tp_x dt$

$E\left[\min(K_x,n)^2\right] = \sum_{k=1}^{n-1} k^2\,{}_kp_x\ q_{x+k} + n^2\,{}_np_x = \sum_{k=1}^{n}(2k-1)\,{}_kp_x$

Variance: $Var(T_x) = E[T_x^2] - E[T_x]^2$

$Var(K_x) = E[K_x^2] - E[K_x]^2$

Recursive formulas: $\overset{o}{e}_x = \overset{o}{e}_{x:\overline{n}|} + {}_np_x\ \overset{o}{e}_{x+n}$

$e_x = e_{x:\overline{n}|} + {}_np_x\ e_{x+n} \quad \rightarrow \quad e_x = p_x + p_x\ e_{x+1}$

$\overset{o}{e}_{x:\overline{n}|} = \overset{o}{e}_{x:\overline{m}|} + {}_mp_x\ \overset{o}{e}_{x+m:\overline{n-m}|}$ for $m < n$

$e_{x:\overline{n}|} = e_{x:\overline{m}|} + {}_mp_x\ e_{x+m:\overline{n-m}|}$ for $m < n \quad \rightarrow \quad e_{x:\overline{n}|} = p_x + p_x\ e_{x+1:\overline{n-1}|}$

C6. MORTALITY LAWS

Gompertz's law: $\mu_x = Bc^x$ for $c > 1$ $\rightarrow$ ${}_tp_x = e^{-\frac{Bc^x\left(c^t-1\right)}{\log c}}$

Makeham's law: $\mu_x = A + Bc^x$ for $c > 1$ $\rightarrow$ ${}_tp_x = e^{-At-\frac{Bc^x\left(c^t-1\right)}{\log c}}$

Weibull distribution: $\mu_x = kx^n$ $\rightarrow$ ${}_tp_x = e^{-\frac{k\left((x+t)^{n+1}-x^{n+1}\right)}{n+1}}$

Exponential distribution: $\mu_x = \mu$ $\rightarrow$ ${}_tp_x = e^{-\mu t}$

(Constant force of mortality)

Uniform distribution: $\mu_x = \frac{1}{\omega-x}$ for $0 \leq x \leq \omega$ $\rightarrow$ ${}_tp_x = 1 - \frac{t}{\omega-x}$

Beta distribution: $\mu_x = \frac{\alpha}{\omega-x}$ for $0 \leq x \leq \omega$ $\rightarrow$ ${}_tp_x = \left(1 - \frac{t}{\omega-x}\right)^{\alpha}$

C7. APPROXIMATIONS

UDD between integral ages: $l_{x+s} = l_x - sd_x$ $\rightarrow$ ${}_sq_x = sq_x$ $\quad {}_sq_{x+t} = \dfrac{Sq_x}{1-tq_x}$ $\quad q_x = {}_sp_x\ \mu_{x+s}$

CFM between integral ages: $l_{x+s} = l_x \times (p_x)^s$ $\rightarrow$ ${}_sp_x = (p_x)^s$ $\quad {}_sp_{x+t} = (p_x)^s$ $\quad \mu_{x+s} = -\log p_x$

These are for $0 \leq s, t \leq 1$ and $0 \leq s + t \leq 1$.

C8. SELECT SURVIVAL MODEL

***k*-year select period:**

$q_{[x]+h} < q_{x+h}$ for $h < k$

$q_{[x]+h} = q_{x+h}$ for $h \geq k$

$p_{[x]+h} > p_{x+h}$ for $h < k$

$p_{[x]+h} = p_{x+h}$ for $h \geq k$

D. ESTIMATION

D1. EMPIRICAL ESTIMATION

For seriatim data: $\hat{S}(t) = \dfrac{\text{Number of survivors at time } t}{n} = \dfrac{n_t}{n}$

Variance: $\text{Var}(\hat{S}(t)) \approx \dfrac{n_t\left(n-n_t\right)}{n^3} = \hat{S}(t)^2\left(\frac{1}{n_t} - \frac{1}{n}\right)$

For grouped data: $\hat{S}(t) = \dfrac{\left(t_U - t\right)\hat{S}\left(t_L\right) + \left(t - t_L\right)\hat{S}\left(t_U\right)}{t_U - t_L}$ This is called ogive empirical survival function.

Where: $t_L \leq t < t_U$

D2. KM AND NA ESTIMATES

Define the following:

$t_{(j)}$ is the time of each event for a mortality study — An event can be an entry, exit, or death.

c_j^L is the number of entries at time $t_{(j)}$ — These are left truncated observations.

c_j^R is the number of exits at time $t_{(j)}$ — These are right censored observations.

d_j is the number of deaths at time $t_{(j)}$ — They are observations with exact values.

$r_j = r_{j-1} + c_{j-1}^L - c_{j-1}^R - d_{j-1}$ is the number of (active) lives at time $t_{(j)}$

Kaplan-Meier estimator: $\hat{S}(t) = \prod_{j:t_{(j)}\le t}\left(1 - \frac{d_j}{r_j}\right)$ — Also called product limit estimator.

Greenwood's formula: $\mathrm{Var}(\hat{S}(t)) \approx (\hat{S}(t))^2 \sum_{j:t_{(j)}\le t}\left(\frac{d_j}{r_j(r_j-d_j)}\right)$

Linear CI for $S(t)$: $\hat{S}(t) \pm z\sqrt{\mathrm{Var}(\hat{S}(t))}$

Log-transformed CI for $S(t)$: $\left(\hat{S}(t)^{\frac{1}{U}},\ \hat{S}(t)^{U}\right)$ where $U = \exp\left(\frac{z\sqrt{\widehat{Var}\left(\hat{S}(t)\right)}}{\hat{S}(t)\log\hat{S}(t)}\right)$

Nelson-Aalen estimator: $\widehat{H}(t) = \sum_{j:t_{(j)}\le t}\left(\frac{d_j}{r_j}\right) \quad\rightarrow\quad \hat{S}(t) = e^{-\hat{H}(t)}$

Klein's formula: $\mathrm{Var}(\widehat{H}(t)) \approx \sum_{j:t_{(j)}\le t}\left(\frac{d_j(r_j-d_j)}{r_j^3}\right) \quad\rightarrow\quad \mathrm{Var}(\hat{S}(t)) \approx (\hat{S}(t))^2 \sum_{j:t_{(j)}\le t}\left(\frac{d_j(r_j-d_j)}{r_j^3}\right)$

Linear CI for $H(t)$: $\widehat{H}(t) \pm z\sqrt{\mathrm{Var}(\widehat{H}(t))}$

Log-transformed CI for $H(t)$: $\left(\frac{\hat{H}(t)}{U},\ \hat{H}(t)\,U\right)$ where $U = \exp\left(\frac{z\sqrt{\widehat{Var}\left(\hat{H}(t)\right)}}{\hat{H}(t)}\right)$

Exponential extrapolation: $\hat{S}(t) = \hat{S}(t_{\max})^{t/t_{\max}}$ for $t \ge t_{\max}$

Where: $t_{\max}$ is the time the study ends or it is the time of the last event.

D3. MAXIMUM LIKELIHOOD ESTIMATES

Central exposed to risk: E_x^c = Total waiting time between ages x and $x+1$

MLE of μ_x: $\hat{\mu}_x = \frac{d_x}{E_x^c}$

Assumption: μ_x is constant between ages x and $x+1$

MLE of q_x: $\hat{q}_x = 1 - e^{-\hat{\mu}_x}$

Variance: $\mathrm{Var}(\hat{q}_x) \approx (1-\hat{q}_x)^2 \frac{d_x}{(E_x^c)^2}$

Assumption: Deaths are uniformly distributed between ages x and $x+1$

Actuarial estimate of q_x: $\tilde{q} = \frac{d_x}{E_x^c + \frac{d_x}{2}}$

E. INSURANCE

E1. ACTUARIAL FUNCTIONS

Endowment:

$${}_nE_x = A_{\underset{x:\overline{n}|}{}\overset{\ 1}{}} = v^n {}_np_x \qquad {}_n^2E_x = {}^2A_{\underset{x:\overline{n}|}{}\overset{\ 1}{}} = (v^n)^2 {}_np_x$$

Insurance (Continuous):

$$\bar{A}_x = \int_0^\infty v^t {}_tp_x\ \mu_{x+t} dt \qquad {}^2\bar{A}_x = \int_0^\infty (v^t)^2 {}_tp_x\ \mu_{x+t} dt$$

$$\bar{A}^{1}_{x:\overline{n}|} = \int_0^n v^t {}_tp_x\ \mu_{x+t} dt \qquad {}^2\bar{A}^{1}_{x:\overline{n}|} = \int_0^n (v^t)^2 {}_tp_x\ \mu_{x+t} dt$$

$$\bar{A}_{x:\overline{n}|} = \int_0^n v^t {}_tp_x\ \mu_{x+t} dt + v^n {}_np_x \qquad {}^2\bar{A}_{x:\overline{n}|} = \int_0^n (v^t)^2 {}_tp_x\ \mu_{x+t} dt + (v^n)^2 {}_np_x$$

Insurance (Discrete):

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x\ q_{x+k} \qquad {}^2A_x = \sum_{k=0}^{\infty} \left(v^{k+1}\right)^2 {}_kp_x\ q_{x+k}$$

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x\ q_{x+k} \qquad {}^2A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \left(v^{k+1}\right)^2 {}_kp_x\ q_{x+k}$$

$$A_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x\ q_{x+k} + v^n {}_np_x \qquad {}^2A_{x:\overline{n}|} = \sum_{k=0}^{n-1} \left(v^{k+1}\right)^2 {}_kp_x\ q_{x+k} + (v^n)^2 {}_np_x$$

Insurance (mthly):

$$A^{(m)\,1}_{x:\overline{n}|} = \sum_{k=0}^{nm-1} v^{k/m+1/m} {}_{k/m}p_x\ {}_{1/m}q_{x+k/m}$$

Relations:

$$\bar{A}_x = \bar{A}^{1}_{x:\overline{n}|} + {}_nE_x\ \bar{A}_{x+n} \qquad A_x = A^{1}_{x:\overline{n}|} + {}_nE_x\ A_{x+n}$$

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + {}_nE_x \qquad A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x$$

$${}_{n|}\bar{A}_x = {}_nE_x\ \bar{A}_{x+n} \qquad {}_{n|}A_x = {}_nE_x A_{x+n}$$

Recursive formulas:

$$A_x = vq_x + vp_x\ A_{x+1} \qquad {}^2A_x = v^2q_x + v^2p_x\ {}^2A_{x+1}$$

$$A^{1}_{x:\overline{n}|} = vq_x + vp_x\ A^{\ 1}_{x+1:\overline{n-1}|} \qquad {}^2A^{1}_{x:\overline{n}|} = v^2q_x + v^2p_x\ {}^2A^{\ 1}_{x+1:\overline{n-1}|}$$

E2. PRESENT VALUES

Policy	Death benefit paid at the moment of death	Death benefit paid at the end of the year of death
Whole life insurance	$Z = bv^{T_x}, T_x > 0$	$Z = bv^{K_x+1}, \quad K_x = 0, 1, 2, \ldots, \infty$
n-year term insurance	$Z = \begin{cases} bv^{T_x}, & T_x < n \\ 0, & T_x \geq n \end{cases}$	$Z = \begin{cases} bv^{K_x+1}, & K_x = 0, 1, 2, \ldots, n-1 \\ 0, & K_x = n, n+1, \ldots, \infty \end{cases}$
n-year endowment insurance	$Z = \begin{cases} bv^{T_x}, & T_x < n \\ bv^n, & T_x \geq n \end{cases}$	$Z = \begin{cases} bv^{K_x+1}, & K_x = 0, 1, 2, \ldots, n-1 \\ bv^n, & K_x = n, n+1, \ldots, \infty \end{cases}$
n-year pure endowment	$Z = \begin{cases} 0, & T_x < n \\ bv^n, & T_x \geq n \end{cases}$	

n-year deferred whole life insurance	$Z = \begin{cases} 0, & T_x < n \\ bv^{T_x}, & T_x \geq n \end{cases}$	$Z = \begin{cases} 0, & K_x = 0, 1, 2, \ldots, n-1 \\ bv^{K_x+1}, & K_x = n, n+1, \ldots, \infty \end{cases}$

E3. EXPECTED PRESENT VALUES

Policy	Death benefit paid at the moment of death	Death benefit paid at the end of the year of death
Whole life insurance	$E[Z] = b\bar{A}_x$	$E[Z] = bA_x$
n-year term insurance	$E[Z] = b\bar{A}^{1}_{x:\overline{n}\vert}$	$E[Z] = bA^{1}_{x:\overline{n}\vert}$
n-year endowment insurance	$E[Z] = b\bar{A}_{x:\overline{n}\vert}$	$E[Z] = bA_{x:\overline{n}\vert}$
n-year pure endowment	$E[Z] = b\,{}_nE_x$	
n-year deferred whole life insurance	$E[Z] = b\,{}_{n\vert}\bar{A}_x$	$E[Z] = b\,{}_{n\vert}A_x$

E4. VARIANCE OF PRESENT VALUES

Policy	Death benefit paid at the moment of death	Death benefit paid at the end of the year of death
Whole life insurance	$Var(Z) = b^2\left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$	$Var(Z) = b^2\left({}^2A_x - (A_x)^2\right)$
n-year term insurance	$Var(Z) = b^2\left({}^2\bar{A}^{1}_{x:\overline{n}\vert} - \left(\bar{A}^{1}_{x:\overline{n}\vert}\right)^2\right)$	$Var(Z) = b^2\left({}^2A^{1}_{x:\overline{n}\vert} - \left(A^{1}_{x:\overline{n}\vert}\right)^2\right)$
n-year endowment insurance	$Var(Z) = b^2\left({}^2\bar{A}_{x:\overline{n}\vert} - (\bar{A}_{x:\overline{n}\vert})^2\right)$	$Var(Z) = b^2\left({}^2A_{x:\overline{n}\vert} - (A_{x:\overline{n}\vert})^2\right)$
n-year pure endowment	$Var(Z) = b^2\left({}^2_nE_x - ({}_nE_x)^2\right)$	
n-year deferred whole life insurance	$Var(Z) = b^2\left({}^2_{n\vert}\bar{A}_x - ({}_{n\vert}\bar{A}_x)^2\right)$	$Var(Z) = b^2\left({}^2_{n\vert}A_x - ({}_{n\vert}A_x)^2\right)$

E5. APPROXIMATIONS

UDD between integral ages:

$\bar{A}_x = \frac{i}{\delta}A_x$ $\quad$ $\bar{A}^{1}_{x:\overline{n}\vert} = \frac{i}{\delta}A^{1}_{x:\overline{n}\vert}$ $\quad$ $\bar{A}_{x:\overline{n}\vert} \neq \frac{i}{\delta}A_{x:\overline{n}\vert}$.

$A_x^{(m)} = \frac{i}{i^{(m)}}A_x$

${}^2\bar{A}_x = \frac{2i+i^2}{2\delta}\ {}^2A_x$

Claims acceleration approach:

$\bar{A}_x = (1+i)^{0.5}\ A_x$ $\quad$ $\bar{A}^{1}_{x:\overline{n}\vert} = (1+i)^{0.5}\ A^{1}_{x:\overline{n}\vert}$ $\quad$ $\bar{A}_{x:\overline{n}\vert} \neq (1+i)^{0.5}\ A_{x:\overline{n}\vert}$.

$A_x^{(m)} = (1+i)^{\frac{m-1}{2m}}\ A_x$

${}^2\bar{A}_x = (1+i)\ {}^2A_x$

F. ANNUITIES

F1. ACTUARIAL FUNCTIONS

Annuity (Continuous):

$\bar{a}_x = \int_0^\infty \left(\frac{1-v^t}{\delta}\right) {}_tp_x\ \mu_{x+t}dt$ $\quad {}^2\bar{a}_x = \int_0^\infty \left(\frac{1-v^{2t}}{2\delta}\right) {}_tp_x\ \mu_{x+t}dt$

$\bar{a}_{x:\overline{n}|} = \int_0^n \left(\frac{1-v^t}{\delta}\right) {}_tp_x\ \mu_{x+t}dt + \left(\frac{1-v^n}{\delta}\right) {}_np_x$ $\quad {}^2\bar{a}_{x:\overline{n}|} = \int_0^n \left(\frac{1-v^{2t}}{2\delta}\right) {}_tp_x\ \mu_{x+t}dt + \left(\frac{1-v^{2n}}{2\delta}\right) {}_np_x$

Annuity (Discrete):

$\ddot{a}_x = \sum_{k=0}^{\infty} \left(\frac{1-v^{k+1}}{d}\right) {}_kp_x\ q_{x+k}$ $\quad {}^2\ddot{a}_x = \sum_{k=0}^{\infty} \left(\frac{1-v^{2k+2}}{2d-d^2}\right) {}_kp_x\ q_{x+k}$

$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \left(\frac{1-v^{k+1}}{d}\right) {}_kp_x\ q_{x+k} + \left(\frac{1-v^n}{d}\right) {}_np_x$ $\quad {}^2\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \left(\frac{1-v^{2k+2}}{2d-d^2}\right) {}_kp_x\ q_{x+k} + \left(\frac{1-v^{2n}}{2d-d^2}\right) {}_np_x$

Simpler formulas:

$\bar{a}_x = \int_0^\infty v^t {}_tp_x dt$ $\quad {}^2\bar{a}_x = \int_0^\infty v^{2t} {}_tp_x dt$

$\bar{a}_{x:\overline{n}|} = \int_0^n v^t {}_tp_x dt$ $\quad {}^2\bar{a}_{x:\overline{n}|} = \int_0^n v^{2t} {}_tp_x dt$

$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x$ $\quad {}^2\ddot{a}_x = \sum_{k=0}^{\infty} v^{2k} {}_kp_x$

$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x$ $\quad {}^2\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{2k} {}_kp_x$

$\ddot{a}^{(m)}_{x:\overline{n}|} = \sum_{k=0}^{nm-1} \frac{1}{m} v^{k/m} {}_{k/m}p_x$

Relations:

$\bar{a}_x = \bar{a}_{x:\overline{n}|} + {}_nE_x\ \bar{a}_{x+n}$ $\quad \ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x\ \ddot{a}_{x+n}$

${}_{n|}\bar{a}_x = {}_nE_x\ \bar{a}_{x+n}$ $\quad {}_{n|}\ddot{a}_x = {}_nE_x\ \ddot{a}_{x+n}$

$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_nE_x\ \bar{a}_{x+n}$ $\quad \ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_nE_x\ \ddot{a}_{x+n}$

$\ddot{a}_x = 1 + a_x$

$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|} = 1 + a_{x:\overline{n}|} - {}_nE_x$

Recursive formulas:

$\ddot{a}_x = 1 + vp_x\ \ddot{a}_{x+1}$ $\quad {}^2\ddot{a}_x = 1 + v^2p_x\ {}^2\ddot{a}_{x+1}$

$\ddot{a}_{x:\overline{n}|} = 1 + vp_x\ \ddot{a}_{x+1:\overline{n-1}|}$ $\quad {}^2\ddot{a}_{x:\overline{n}|} = 1 + v^2p_x\ {}^2\ddot{a}_{x+1:\overline{n-1}|}$

Insurance to annuity:

$\bar{a}_x = \frac{1-\bar{A}_x}{\delta}$ $\quad {}^2\bar{a}_x = \frac{1-{}^2\bar{A}_x}{2\delta}$

$\bar{a}_{x:\overline{n}|} = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta}$ $\quad {}^2\bar{a}_{x:\overline{n}|} = \frac{1-{}^2\bar{A}_{x:\overline{n}|}}{2\delta}$

$\ddot{a}_x = \frac{1-A_x}{d}$ $\quad {}^2\ddot{a}_x = \frac{1-{}^2A_x}{2d-d^2}$

$\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$ $\quad {}^2\ddot{a}_{x:\overline{n}|} = \frac{1-{}^2A_{x:\overline{n}|}}{2d-d^2}$

$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$

F2. PRESENT VALUES

Policy	Payments made continuously	Payments at the beginning of each year
Whole life annuity	$Y = b\left(\frac{1-v^{T_x}}{\delta}\right), T_x > 0$	$Y = b\left(\frac{1-v^{K_x+1}}{d}\right), \quad K_x = 0,1,2,\ldots,\infty$
n-year term annuity	$Y = \begin{cases} b\left(\frac{1-v^{T_x}}{\delta}\right), & T_x < n \\ b\left(\frac{1-v^{n}}{\delta}\right), & T_x \geq n \end{cases}$	$Y = \begin{cases} b\left(\frac{1-v^{K_x+1}}{d}\right), & K_x = 0,1,2,\ldots,n-1 \\ b\left(\frac{1-v^{n}}{d}\right), & K_x = n,n+1,\ldots,\infty \end{cases}$
n-year certain whole life annuity	$Y = \begin{cases} b\left(\frac{1-v^{n}}{\delta}\right), & T_x < n \\ b\left(\frac{1-v^{T_x}}{\delta}\right), & T_x \geq n \end{cases}$	$Y = \begin{cases} b\left(\frac{1-v^{n}}{d}\right), & K_x = 0,1,2,\ldots,n-1 \\ b\left(\frac{1-v^{K_x+1}}{d}\right), & K_x = n,n+1,\ldots,\infty \end{cases}$
n-year deferred whole life annuity	$Y = \begin{cases} 0, & T_x < n \\ b\left(\frac{v^n-v^{T_x}}{\delta}\right), & T_x \geq n \end{cases}$	$Y = \begin{cases} 0, & K_x = 0,1,2,\ldots,n-1 \\ b\left(\frac{v^n-v^{K_x+1}}{d}\right), & K_x = n,n+1,\ldots,\infty \end{cases}$

F3. EXPECTED PRESENT VALUES

Policy	Payments made continuously	Payments at the beginning of each year
Whole life annuity	$E[Y] = b\bar{a}_x$	$E[Y] = b\ddot{a}_x$
n-year term annuity	$E[Y] = b\bar{a}_{x:\overline{n}\rvert}$	$E[Y] = b\ddot{a}_{x:\overline{n}\rvert}$
n-year certain whole life annuity	$E[Y] = b\bar{a}_{\overline{x:\overline{n}\rvert}}$	$E[Y] = b\ddot{a}_{\overline{x:\overline{n}\rvert}}$
n-year deferred whole life annuity	$E[Y] = b_{n\rvert}\bar{a}_x$	$E[Y] = b_{n\rvert}\ddot{a}_x$

F4. VARIANCE OF PRESENT VALUES

Policy	Payments made continuously	Payments at the beginning of each year
Whole life annuity	$Var(Y) = b^2\frac{{}^2\bar{A}_x-(\bar{A}_x)^2}{\delta^2}$	$Var(Y) = b^2\frac{{}^2A_x-(A_x)^2}{d^2}$
n-year term annuity	$Var(Y) = b^2\frac{{}^2\bar{A}_{x:\overline{n}\rvert}-(\bar{A}_{x:\overline{n}\rvert})^2}{\delta^2}$	$Var(Y) = b^2\frac{{}^2A_{x:\overline{n}\rvert}-(A_{x:\overline{n}\rvert})^2}{d^2}$

F5. APPROXIMATIONS

UDD between integral ages: $\bar{a}_x = \frac{id}{\delta^2}\ddot{a}_x - \frac{i-\delta}{\delta^2}$

$\ddot{a}_x^{(m)} = \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}$

$\ddot{a}_{x:\overline{n}|}^{(m)} = \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_{x:\overline{n}|} - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}\left(1 - {}_nE_x\right)$

Woolhouse formula, 2 terms: $\bar{a}_x \approx \ddot{a}_x - \frac{1}{2}$

$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$

$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - {}_nE_x\right)$

Woolhouse formula, 3 terms: $\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}\left(\mu_x + \delta\right)$

$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left(\mu_x + \delta\right)$ where $\mu_x \approx -\frac{1}{2}\left(\log p_{x-1} + \log p_x\right)$ if not given.

$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - {}_nE_x\right) - \frac{m^2-1}{12m^2}\left(\mu_x + \delta - {}_nE_x\left(\mu_{x+n} + \delta\right)\right)$

G. PREMIUMS

G1. ACTUARIAL FUNCTIONS

Premium (Discrete): $P_x = \frac{A_x}{\ddot{a}_x}$ $\quad P_{\overset{1}{x}:\overline{n}|} = \frac{A_{\overset{1}{x}:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$ $\quad P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$

Premium (Continuous): $\bar{P}\left(\bar{A}_x\right) = \frac{\bar{A}_x}{\bar{a}_x}$ $\quad \bar{P}\left(\bar{A}_{\overset{1}{x}:\overline{n}|}\right) = \frac{\bar{A}_{\overset{1}{x}:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$ $\quad \bar{P}\left(\bar{A}_{x:\overline{n}|}\right) = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$

G2. EQUIVALENCE PRINCIPLE

Loss-at-issue: ${}_0L^n = PV(\text{Benefits}) - PV(\text{Premiums})$

${}_0L^g = PV(\text{Benefits}) + PV(\text{Expenses}) - PV(\text{Premiums})$

Equivalence principle: $\mathrm{E}[{}_0L] = EPV(\text{Benefits}) - EPV(\text{Premiums}) = 0$

$\mathrm{E}[{}_0L^g] = EPV(\text{Benefits}) + EPV(\text{Expenses}) - EPV(\text{Premiums}) = 0$

G3. EXPECTATION AND VARIANCE

Fully continuous whole life insurance	Fully discrete whole life insurance
${}_0L^g = (b+E)\,v^{T_x} - (G-e)\left(\frac{1-v^{T_x}}{\delta}\right), \quad T_x \geq 0$ ${}_0L^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$	${}_0L^g = (b+E)\,v^{K_x+1} - (G-e)\left(\frac{1-v^{K_x+1}}{d}\right), \quad K_x = 0,1,2,\ldots,\infty$ ${}_0L^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$
$E\left[{}_0L^g\right] = (b+E)\,\bar{A}_x - (G-e)\,\bar{a}_x$ $Var\left({}_0L^g\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right)$	$E\left[{}_0L^g\right] = (b+E)\,A_x - (G-e)\,\ddot{a}_x$ $Var\left({}_0L^g\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_x - \left(A_x\right)^2\right)$

Fully continuous n-year endowment insurance	Fully discrete n-year endowment insurance
${}_0L^g = \begin{cases} (b+E)\,v^{T_x} - (G-e)\left(\frac{1-v^{T_x}}{\delta}\right), & T_x < n \\ (b+E)\,v^{n} - (G-e)\left(\frac{1-v^{n}}{\delta}\right), & T_x \geq n \end{cases}$	${}_0L^g = \begin{cases} (b+E)\,v^{K_x+1} - (G-e)\left(\frac{1-v^{K_x+1}}{d}\right), & K_x = 0,1,\ldots,n-1 \\ (b+E)\,v^{n} - (G-e)\left(\frac{1-v^{n}}{d}\right), & K_x = n,n+1,\ldots,\infty \end{cases}$
${}_0L^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$	${}_0L^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$
$E\left[{}_0L^g\right] = (b+E)\,\bar{A}_{x:\overline{n}\rvert} - (G-e)\,\bar{a}_{x:\overline{n}\rvert}$	$E\left[{}_0L^g\right] = (b+E)\,A_{x:\overline{n}\rvert} - (G-e)\,\ddot{a}_{x:\overline{n}\rvert}$
$Var\left({}_0L^g\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_{x:\overline{n}\rvert} - \left(\bar{A}_{x:\overline{n}\rvert}\right)^2\right)$	$Var\left({}_0L^g\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_{x:\overline{n}\rvert} - \left(A_{x:\overline{n}\rvert}\right)^2\right)$

G4. PORTFOLIO PERCENTILE PRINCIPLE

Aggregate losses: $S = L_1^g + L_2^g + \cdots + L_n^g \dot{\sim} N\left(\mu, \sigma^2\right)$

Mean: $\mu = E[S] = nE\left[{}_0L^g\right]$

Variance: $\sigma^2 = \text{Var}(S) = n\,\text{Var}\left({}_0L^g\right)$

Find premium such that: $\Pr(S<0) \approx N\left(\frac{0-nE[{}_0L^g]}{\sqrt{n\,\text{Var}({}_0L^g)}}\right) = \alpha \quad \rightarrow \quad \frac{0-nE[{}_0L^g]}{\sqrt{n\,\text{Var}({}_0L^g)}} = z_\alpha$

H. RESERVES

H1. PROSPECTIVE FORMULA

Net premium reserve: ${}_tV^n = E\left[{}_tL \mid T_x \geq t\right] = EPV_t\text{ (Benefits)} - EPV_t\text{ (Net Premiums)}$

Gross premium reserve: ${}_tV^g = E\left[{}_tL^g \mid T_x \geq t\right] = EPV_t\text{ (Benefits)} + EPV_t\text{ (Expenses)}$
$- EPV_t\text{ (Gross Premiums)}$

Expense reserve: ${}_tV^e = {}_tV^g - {}_tV^n = EPV_t\text{(Expenses)} - EPV_t\text{(Expense Loadings)}$

H2. EXPECTATION AND VARIANCE

Fully continuous whole life insurance	Fully discrete whole life insurance
${}_tL^g = (b+E)\,v^{T_{x+t}} - (G-e)\left(\frac{1-v^{T_{x+t}}}{\delta}\right), \quad T_{x+t} \geq 0$	${}_kL^g = (b+E)\,v^{K_{x+k}+1} - (G-e)\left(\frac{1-v^{K_{x+k}+1}}{d}\right),$ $K_{x+k} = 0,1,2,\ldots,\infty$
${}_tL^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$	${}_kL^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$
$E\left[{}_tL^g \mid T_x \geq t\right] = (b+E)\bar{A}_{x+t} - (G-e)\bar{a}_{x+t}$	$E\left[{}_kL^g \mid K_x \geq k\right] = (b+E)A_{x+k} - (G-e)\ddot{a}_{x+k}$
$Var\left({}_tL^g \mid T_x \geq t\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_{x+t} - \left(\bar{A}_{x+t}\right)^2\right)$	$Var\left({}_kL^g \mid K_x \geq k\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_{x+k} - \left(A_{x+k}\right)^2\right)$

Fully continuous n-year endowment insurance	Fully discrete n-year endowment insurance
${}_tL^g = \begin{cases} (b+E)\,v^{T_{x+t}} - (G-e)\left(\frac{1-v^{T_{x+t}}}{\delta}\right), & T_{x+t} < n-t \\ (b+E)\,v^{n-t} - (G-e)\left(\frac{1-v^{n-t}}{\delta}\right), & T_{x+t} \ge n-t \end{cases}$	${}_kL^g = \begin{cases} (b+E)\,v^{K_{x+k}+1} - (G-e)\left(\frac{1-v^{K_{x+k}+1}}{d}\right), & K_{x+k} = 0,\ldots,n-k-1 \\ (b+E)\,v^{n-k} - (G-e)\left(\frac{1-v^{n-k}}{d}\right), & K_{x+k} = n-k, n-k+1,\ldots \end{cases}$
${}_tL^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$	${}_kL^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$
$E\left[{}_tL^g \mid T_x \ge t\right] = (b+E)\bar{A}_{x+t:\overline{n-t}\|} - (G-e)\bar{a}_{x+t:\overline{n-t}\|}$	$E\left[{}_kL^g \mid K_x \ge k\right] = (b+E)A_{x+k:\overline{n-k}\|} - (G-e)\ddot{a}_{x+k:\overline{n-k}\|}$
$Var\left({}_tL^g \mid T_x \ge t\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_{x+t:\overline{n-t}\|} - \left(\bar{A}_{x+t:\overline{n-t}\|}\right)^2\right)$	$Var\left({}_kL^g \mid K_x \ge k\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_{x+k:\overline{n-k}\|} - \left(A_{x+k:\overline{n-k}\|}\right)^2\right)$

H3. RECURSIVE FORMULA

Net premium reserve: $({}_kV + P)(1+i) = bq_{x+k} + {}_{k+1}V\ p_{x+k}$

Gross premium Reserve: $({}_kV^g + G - e)(1+i) = (b+E)\,q_{x+k} + {}_{k+1}V^g\ p_{x+k}$

For reserves between premium dates: $({}_kV + P)(1+i_k)^s = b\,v^{1-s}\ {}_sq_{x+k} + {}_{k+s}V\ {}_sp_{x+k}$ where $0 < s < 1$.

${}_{k+s}V\,(1+i_k)^{1-s} = b\ {}_{1-s}q_{x+k+s} + {}_{k+1}V\ {}_{1-s}p_{x+k+s}$ where $0 < s < 1$.

For a special policy that pays $FA + {}_kV$ upon death: $({}_{k-1}V + P)(1+i) = (FA + {}_kV)\ q_{x+k-1} + {}_kV\ p_{x+k-1}$

Rearrange: ${}_kV = ({}_{k-1}V + P)(1+i) - FA\ q_{x+k-1}$

Obtain the formula: ${}_kV = P\ddot{a}_{\overline{k}|}(1+i)^k - FA\sum_{j=1}^{k} q_{x+j-1}(1+i)^{k-j}$

H4. FPT RESERVES

First year premium: $E\left[{}_0L^{FPT}\right] = vq_x - \alpha = 0 \quad\rightarrow\quad \alpha = vq_x$

Renewal year premium:

${}_1V^{FPT} = A_{x+1} - \beta\ddot{a}_{x+1} = 0 \quad\rightarrow\quad \beta = \frac{A_{x+1}}{\ddot{a}_{x+1}}$ whole life

${}_1V^{FPT} = A^{\ 1}_{x+1:\overline{n-1}|} - \beta\ddot{a}_{x+1:\overline{n-1}|} = 0 \quad\rightarrow\quad \beta = \frac{A^{\ 1}_{x+1:\overline{n-1}|}}{\ddot{a}_{x+1:\overline{n-1}|}}$ n-year term

${}_1V^{FPT} = A_{x+1:\overline{n-1}|} - \beta\ddot{a}_{x+1:\overline{n-1}|} = 0 \quad\rightarrow\quad \beta = \frac{A_{x+1:\overline{n-1}|}}{\ddot{a}_{x+1:\overline{n-1}|}}$ n-year endowment