UTILITY THEORY

Utility function $U(w)$

Non-satiation: $U'(w) > 0$ | **Risk averse:** $U''(w) < 0$

Risk neutral: $U''(w) = 0$ | **Risk seeking:** $U''(w) > 0$

	Absolute risk aversion	Relative risk aversion
Formula	$A(w) = -U''(w)/U'(w)$	$R(w) = -wU''(w)/U'(w)$
Increasing	$A'(w) > 0$	$R'(w) > 0$
Constant	$A'(w) = 0$	$R'(w) = 0$
Decreasing	$A'(w) < 0$	$R'(w) < 0$

Maximum premium P — policy holder for random loss X with initial wealth a: $\mathrm{E}[U(a-X)] = U(a-P)$

Minimum premium Q — insurer for random loss Y with initial wealth a: $\mathrm{E}[U(a+Q-Y)] = U(a)$

RISK MEASURES

	Continuous variable	Discrete variable
Variance	$\int_{-\infty}^{\infty} (\mu - x)^2 f(x)\, dx$	$\sum_i (\mu - x_i)^2 p_i$ for all i
Semi-Variance	$\int_{-\infty}^{\mu} (\mu - x)^2 f(x)\, dx$	$\sum_i (\mu - x_i)^2 p_i$ for i: $x_i < \mu$

If the distribution is symmetric, Semi-variance $= \frac{1}{2}\times$ Variance

Value-at-Risk (VaR) at p: $\mathrm{VaR}_p = t$ where t is the $100p$-th percentile, i.e. $P(X < t) = p$.
For Normal distribution: $\mathrm{VaR}_p = \mu + Z_p\sigma$

Shortfall probability at L: $P(X < L) = \int_{-\infty}^{L} f(x)\, dx$ | $P(X < L) = \sum_i p_i$ for i: $x_i < L$

Expected Shortfall at L: $\mathrm{E}\left[\max\{0, L - x\}\right] = \int_{-\infty}^{L} (L - x) f(x)\, dx$

$\mathrm{E}\left[\max\{0, L - x\}\right] = \sum_i (L - x_i) p_i$ for i: $x_i < L$

ASSET VALUATIONS

Notations

$E_i = E(R_i)$ $\quad V_i = \sigma_i^2 = \mathrm{Var}(R_i)$ $\quad C_{ij} = \mathrm{Cov}(R_i, R_j)$ $\quad \rho_{ij} = \mathrm{Corr}(R_i, R_j) = \dfrac{C_{ij}}{\sigma_i \sigma_j}$

Portfolio P of N Assets $\quad E_P = \sum_{i=1}^{N} x_i E_i$ $\quad V_P = \sum_{i=1}^{N} x_i^2 V_i + 2\sum_{i=1}^{N}\sum_{j<i} x_i x_j C_{ij}$ $\quad V_p = \sum_i \sum_j x_i x_j C_{ij}$

Minimum Variance Portfolio

Two Assets with $\rho_{12} \neq 1$: $x_1 = \dfrac{V_2 - C_{12}}{V_1 + V_2 - 2C_{12}}$ $\quad$ If $\rho_{12} = \pm 1$: $x_1 = \mp \dfrac{\sigma_2}{\sigma_1 \mp \sigma_2}$

- With n-risky assets $\quad$ Lagrangian: $F(\mathbf{x}, \lambda) = \sum_{ij} x_i C_{ij} x_j - \lambda \left(\sum_i x_i - 1 \right)$ gives equations

$2\sum_j C_{ij} x_j - \lambda = 0 \qquad \sum_i x_i = 1$

Efficient Portfolio

Lagrangian given E_P: $F(\mathbf{x},\lambda) = V - \lambda(E - E_P) + \mu\left(\sum_i x_i - 1\right)$ where $V = \sum_{ij} x_i C_{ij} x_j$

$$2\sum_j C_{ij}x_j - \lambda E_i - \mu = 0$$

$\sum_i E_i x_i = E_P$ $\quad \sum_i x_i = 1$ $\quad$ The solution $\mathbf{x}$ is linear in E_P

Portfolio Diversification

Given $x_i = \frac{1}{n}$: $\quad V_P = \frac{\overline{V}}{n} + \frac{n-1}{n}\overline{C}$ where $\overline{V} = \text{avg}_i[V_i]$, $\overline{C} = \text{avg}_{i \neq j}[C_{ij}]$

CAPM

Risk premium $= E_P - r_f$ $\quad$ Market price of risk $= \phi_P = \frac{E_P - r_f}{\sigma_P}$ $\quad \phi_M = \max_P \phi_P$

For P on the capital market line $\quad E_P - r = \frac{E_M - r}{\sigma_M} \cdot \sigma_P$

For any security i: $\quad E_i - r = \beta_i \cdot (E_M - r)$ where $\beta_i = \frac{\text{Cov}(R_i, R_M)}{V_M}$

Single-index model

$R_i = \alpha_i + \beta_i R_M + \varepsilon_i$ $\quad$ R_M and ε_i are uncorrelated.

ε_i and ε_j are independent. $\quad E_i = \alpha_i + \beta_i \cdot E_M$

$V_i = \beta_i^2 \cdot V_M + V_{\varepsilon_i}$ $\quad C_{ij} = \beta_i \cdot \beta_j \cdot V_M$

Multi-factor model

$$R_i = \alpha_i + \beta_{i1}I_1 + \cdots + \beta_{in}I_n + \varepsilon_i$$

$$E_i = \alpha_i + \sum_j \beta_{ij}\text{E}[I_j]$$

$$V_i = \sum_j \beta_{ij}^2 \text{Var}[I_j] + 2\sum_{j<k} \beta_{ij}\beta_{ik}\text{Cov}(I_j, I_k) + V_{\varepsilon_i}$$

$$C_{ij} = 2\sum_{i<j} \beta_{ik}\beta_{jk}\text{Var}[I_k]$$

STOCHASTIC CALCULUS

Martingale process X_t

Filtration $\{\mathcal{F}_t\}_{t \geq 0}$ $\quad \text{E}[X_s|\mathcal{F}_t] = X_t$ if $t \leq s$ $\quad \text{E}[|X_t|] < \infty$

Supermartingale: $\text{E}[X_s|\mathcal{F}_t] \leq X_t$ $\quad$ Submartingale: $\text{E}[X_s|\mathcal{F}_t] \geq X_t$

Wiener process W_t

$W_0 = 0$ $\quad W_t - W_s \sim N(0, t-s)$ $\quad \text{Cov}[W_s, W_t] = \min\{s, t\}$

Disjoint increments are independent: i.e. $W_{t_2} - W_{t_1}$ and $W_{t_3} - W_{t_2}$ are independent if $t_1 < t_2 < t_3$

Itô integral

$$\int_0^T \sigma(W_t, t)\,dW_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} \sigma(W_{t_i}, t_i)(W_{t_{i+1}} - W_{t_i})$$

if $\sigma(\cdot, t) \in C^2$ and $\text{E}\left[\int_0^T \sigma^2(W_t, dt)\,dt\right] < \infty$

- Itô isometry

$$\text{E}\left[\int_0^T \sigma(W_t, dt)\,dW_t\right] = 0$$

$$\text{Var}\left[\int_0^T \sigma(W_t, dt)\,dW_t\right] = \int_0^T \text{E}\left[\sigma^2(W_t, dt)\right]\,dt$$

- Martingale

$$\text{E}\left[\int_0^T \sigma(W_t, t)\,dW_t \middle| \mathcal{F}_S\right] = \int_0^S \sigma(W_t, t)\,dW_t \text{ if } T > S$$

- Linearity

$$\int_0^T (\sigma(W_t, t) + \nu(W_t, t))\,dW_t = \int_0^T \sigma(W_t, t)\,dW_t + \int_0^T \nu(W_t, t)\,dW_t$$

- Deterministic integrand

$\text{E}\left[\int_0^T \sigma(t)\,dW_t\right] = 0$ $\quad \text{Var}\left[\int_0^T \sigma(t)\,dW_t\right] = \int_0^T \sigma^2(t)\,dt$

Itô Process

$X_T = X_0 + \int_0^T \mu(X_t, t)\,dt + \int_0^T \sigma(X_t, t)\,dW_t$ $\quad dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dW_t$

ITÔ'S LEMMA & SDE

Itô's lemma

Assuming $f(x,t) \in C_x^2 \cap C_t^1$ $\quad dX_t = \mu_t\,dt + \sigma_t\,dW_t$

Using $(dW_t)^2 = dt$, $dW_t\,dt = 0$, $dt\,dW_t = 0$, $(dt)^2 = 0$:

$$df(X_t,t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 = \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t\frac{\partial f}{\partial x}dW_t$$

Arithmetic Brownian motion

$dX_t = \mu\,dt + \sigma\,dW_t$ $\quad X_T = X_0 + \mu T + \sigma W_T \sim N(X_0 + \mu T, \sigma^2 T)$

Geometric Brownian motion

$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$

Apply Itô's lemma with $f(x,t) = e^x$, $dX_t = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dW_t$

$$S_T = S_0\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right] \sim LN\left(\ln X_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

Ornstein-Uhlenbeck process

$dX_t = -\kappa X_t\,dt + \sigma\,dW_t$

Apply Itô's lemma with $f(x,t) = xe^{\kappa t}$

$$X_T = X_0e^{-\kappa T} + \sigma\int_0^T e^{-\kappa(T-t)}\,dW_t \sim N\left(X_0e^{-\kappa T}, \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa T}\right)\right)$$

Mean-reverting process

$dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dW_t$

Apply Itô's lemma with $f(x,t) = xe^{\kappa t}$

$$X_T = X_0e^{-\kappa T} + \theta\left(1 - e^{-\kappa T}\right) + \sigma\int_0^T e^{-\kappa(T-t)}\,dW_t \sim N\left(X_0e^{-\kappa T} + \theta\left(1 - e^{-\kappa T}\right), \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa T}\right)\right)$$

Feller (CIR) process

$dX_t = \kappa(\theta - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t$

INTEREST RATE MODELS

Interest rates

$R(t,T)$ = effective interest rate, $r(t,T)$ = continuous rate,
$P(t,T)$ = discount bond price at t with expiry T

$$P(t,T) = \frac{1}{(1 + R(t,T))^{T-t}} = \exp\left[-(T-t)r(t,T)\right]$$

Short rate: $r_t = r(t, t+\delta) \approx R(t, t+\delta)$

Forward rates

Discrete: $F(0;t,T) = \left(\frac{P(0,t)}{P(0,T)}\right)^{\frac{1}{T-t}} - 1$ $\quad$ Continuous: $f(0;t,T) = r(0,t) + \frac{(r(0,T) - r(0,t))T}{T-t}$

Instantaneous forward rates

$\lim_{t\to T^-} f(0;t,T) = -\frac{\partial}{\partial T}\ln P(0,T)$ $\quad P(t,T) = \exp\left[-\int_t^T f(s,u)\,du\right]$

Short-rate model

$dr_t = \mu(t, r_t)\,dt + \sigma(t, r_t)\,d\hat{W}_t$ $\quad$ under Martingale measure $\mathbf{Q}$

If $P(t,T,r_t) = g(t,r_t)$: g must satisfy:

$$g(T,r) = 1 \text{ and } \frac{\partial g}{\partial t} + \frac{\partial g}{\partial r}\mu(t,r_t) + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}\sigma^2(t,r_t) - r_t g(t,r_t) = 0$$

Vasicek's model

$dr_t = \kappa(\theta - r_t)\,dt + \sigma d\hat{W}_t$ $\quad \theta$: mean level of short rate, κ: speed of reversion

$g(t,r_t) = P(t,T;r_t)$: g must satisfy $g(T,r) = 1$ and $\frac{\partial g}{\partial t} + \frac{\partial g}{\partial r}\kappa(\theta - r_t) + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}\sigma^2 - r_t g(t,r_t) = 0$

$$P(t,T) = \mathrm{E}\left[\exp\left(\int_t^T r_s\,ds\right)\right] = \exp[H(T-t) - G(T-t)r_t] \text{ where:}$$

$$G(\tau) = \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right) \qquad H(\tau) = \left(\theta - \frac{\sigma^2}{\kappa^2}\right)[G(\tau) - \tau] - \frac{\sigma^2}{4\kappa}G^2(\tau)$$

CIR model

$dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,d\hat{W}_t$ $\quad \frac{\partial g}{\partial t} + \frac{\partial g}{\partial r}\kappa(\theta - r_t) + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}\sigma^2 r_t - r_t g(t,r_t) = 0$

$P(t,T) = \exp\left[H(T-t) - G(T-t)r_t\right]$ where $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$ and

$$G(\tau) = \frac{2\left(e^{\gamma\tau}-1\right)}{(\gamma+\kappa)\left(e^{\gamma\tau}-1\right)+2\gamma} \qquad H(\tau) = \frac{2\kappa\theta}{\sigma^2}\ln\left(\frac{2\gamma\exp\left[\frac{1}{2}(\gamma+\kappa)\tau\right]}{(\gamma+\kappa)\left(e^{\gamma\tau}-1\right)+2\gamma}\right)$$

Hull-White model $dr_t = \kappa(t)(\theta(t) - r_t)\,dt + \sigma\,d\hat{W}_t$ Same as Vasicek except $G(T-t)$, $H(T-t)$ replaced with:

$$G(t,T) = \int_t^T \exp\left[-\int_t^s \kappa(u)\,du\right] ds \qquad H(t,T) = -\int_t^T \left(\kappa(s)\theta(s)G(s,T) - \frac{1}{2}\sigma^2 G^2(s,T)\right) ds$$

RISK MODELS

Credit risk Expected Credit Loss (ECL) = Exposure At Default (EAD) × Probability of Default (PD) ×Loss Given Default (LGD)

Recovery rate $= 100\% - LGD$Provision $= \text{PV}[ECL]$

Firm valuation V_t: firm's value, D_t: debt, E_t: equity $V_t = E_t + D_t$

Merton's model If $D_t = D$ $E_T = \max\{V_T - D, 0\}$ $E_t = V_t\Phi(d_1) - De^{-r(T-t)}\Phi(d_2)$

Implicit equation for σ_V: $\sigma_V = \dfrac{\sigma_E E_t}{E_t + De^{-r(T-t)}\Phi(d_2)}$ $d_1 = \dfrac{\ln\frac{V_t}{D} + (r \pm \sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}}$

$1 - \Phi(d_2) = \Phi(-d_2)$ $d_2 = d_1 - \sigma_V\sqrt{T-t}$

Risk-neutral PD$= \Phi(-d_2)$ For real world probabilities, replace μ_V with r when calculating d_1 and d_2

Poisson model, no recovery L: Loan's value if no default λ: Rate of default (hazard rate) $\text{E[loan's value]} = Le^{-\lambda T}$

Model with recovery $q(t,T)$: PD at t expiring at T δ = recovery rate $P(t,T)$ = Loan's value at t if no default

$\text{E[loan's value]} = P(t,T)[q(t,T)\delta + (1-q(t,T))]$ $q(t,T) = \dfrac{1}{1-\delta}\left(1 - \dfrac{\text{E[loan's value]}}{P(t,T)}\right)$

Jarrow-Lando-Turnbull model $Q(t,T) = k \times k$ credit-rating transition matrix with default as the k-th state

Λ = hazard rate matrix

Diagonalisable: $\Lambda = \Sigma\,D\,\Sigma^{-1}$ $D = \text{Diag}[d_j]$, $\Sigma = [\sigma_{ij}]$, $\Sigma^{-1} = [\hat{\sigma}_{ij}]$

$$Q(t,T) = \Sigma e^{D(T-t)}\Sigma^{-1} \qquad q_{ik}(t,T) = \sum_{j=1}^{k-1}\sigma_{ij}\hat{\sigma}_{jk}\left(e^{d_j(T-t)} - 1\right)$$

If $\Lambda = \Lambda(t) = \Sigma D U(t)\Sigma^{-1}$: $q_{ik}(t,T)|\,U(t) = \sum_{j=1}^{k-1}\sigma_{ij}\hat{\sigma}_{jk}\left(\text{E}\left[\exp\left[d_j\int_t^T U(s)\,ds\right]\right] - 1\right)$

LIABILITY VALUATIONS

General loss development model $C_{ij} = r_j \cdot s_i \cdot x_{i+j} + e_{ij}$ C_{ij}: incremental claim in AY_i-DY_j

r_j: development factor (df) for DY_j

s_i: exposure parameter for AY_i

x_{i+j}: parameter for CY_{i+j} e_{ij}: error

Development factors Arithmetic avg: $f_j = \dfrac{1}{n}\sum_{i=1}^{n} f_{ij}$

Weighted avg: $f_j = \dfrac{\sum_i f_{ij} \times L_{i,j-1}}{\sum_i L_{i,j-1}}$ $f_{ij} = \dfrac{L_{i,j}}{L_{i,j-1}}$ where $L_{i,j}$ is cumulative

Chain-ladder (CL) $C_{ij} = r_j \cdot s_i + e_i$ Estimated ultimate claims$_i$ $= L_{ij} \times f_{ult}$ where $f_{ult} = f_j \times f_{j+1} \times \cdots$

Inflation adjusted CL $C_{ij} = r_j \cdot s_i \cdot x_{i+j} + e_{ij}$ To adjust for inflation, first convert cumulative figures to incremental figures

Average cost per claim (ACpC) (1) Calculate past ACpC (2) Develop average ACpC (3) Develop claim count

Estimated ultimate claims$_i$ = Ultimate ACpC$_i$ × Ultimate claim count$_i$

Bornhuetter-Ferguson method Estimated ultimate loss = Earned premium × Loss Ratio;

Estimated reserve = Estimated ultimate loss $\times(1 - 1/f_{ult})$

RUIN THEORY

Aggregate claim $S(t) = \sum_{i=1}^{N(t)} X_i$ $N(t)$: number of claims, X_i: amount for i-th claim, $S(t)$: aggregate claim

Surplus process $U(t) = U + ct - S(t)$ U: initial surplus c: premium income rate

Ruin probability $\Psi(U) = P[U(t) < 0 \text{ for some } t : 0 < t < \infty]$, $\Psi(U,t) = P[U(\tau) < 0 \text{ for some } t : 0 < \tau \leq t]$

$\Psi(U), \Psi(U,\tau)$ are decreasing in U, increasing in τ, $\Psi(U,\tau) < \Psi(U)$ $\lim_{t\to\infty} \Psi(U,t) = \Psi(U)$

Poisson process $N(t)$: Poisson process with rate λ $P[N(t) = k] = \exp[-\lambda t]\dfrac{(\lambda t)^k}{k!}$

Inter-event time: $P(T_k > t) = \exp[-\lambda t]$

Lundberg's inequality $\Psi(U) \leq \exp[-RU]$ R: adjustment coefficient For large U, $\Psi(U) \approx \exp[-RU]$

Compound Poisson model $\mathrm{E}[S(t)] = \lambda t \mathrm{E}[X_i]$ $\mathrm{Var}[S(t)] = \lambda t \mathrm{E}[X_i^2]$ $M_S(r) = \exp[\lambda t(M_X(r) - 1)]$

To make $\mathrm{E}[U(t)] > U$: $c = (1+\theta)\lambda \mathrm{E}[X_i]$ θ: premium loading factor $(\theta > 0)$

R is the unique positive root of: $\lambda(M_X(r) - 1) - cr = 0$ If $X_i \sim \mathrm{Exp}(\alpha)$: $R = \alpha - \dfrac{\lambda}{c}$

Bounds for adjustment coefficient: $R < \dfrac{2(c - \lambda \mathrm{E}[X_i])}{\lambda \mathrm{E}[X_i^2]}$ If $X_i \leq M$: $R > \dfrac{1}{M} \ln \dfrac{c}{\lambda \mathrm{E}[X_i]}$

General aggregate model $c > \mathrm{E}[S_i]$ S_i: aggregate claim in year i $\gamma > 0$: $\lim_{r\to\gamma^-} \mathrm{E}[\exp[r(S_i - c)]] = \infty$

R is unique positive value satisfying: $\mathrm{E}[\exp[R(S_i - c)]] = 1$

Ruin probability dependence $\Psi(U,\tau)$ decreases for larger θ For compound Poisson models: $\Psi(U,\tau)$ increases for larger λ

If $X_i \sim \mathrm{Exp}(1)$ $\Psi(U) = \dfrac{1}{1+\theta}\exp\left(-\dfrac{\theta U}{1+\theta}\right)$

Proportional reinsurance $c = [(1+\theta) - (1+\xi)(1-\alpha)]\lambda \mathrm{E}[X_i]$

ξ: reinsurer premium loading factor α: retention level

Reinsurer constraint: $\alpha > \dfrac{\xi - \theta}{1+\xi}$ Primary insurer constraint: $\alpha > 1 - \dfrac{\theta}{\xi}$ $\xi > \theta$

If $\theta = \xi$, then $\alpha > 0$. i.e. any retention level is possible.

Excess of loss reinsurance $c = (1+\xi)\lambda \mathrm{E}[Y_i] - (1+\theta)\lambda \mathrm{E}[Z_i]$

Reinsurer loss:	$Z_i = \max\{0, X_i - M\}$
Primary insurer loss:	$Y_i = \min\{X, M\}$
R satisfies:	$\lambda + cR = \lambda\left(\int_0^M e^{Rx} f_X(x)\,dx + e^{RM}(1 - F_X(M))\right)$
	M = retention level

OPTION THEORY

Notations

S_t / B_t : share/bond price K: strike T: expiry ϕ_t / ψ_t: number of shares/bond

c_t / p_t: European call/put price

C_t / P_t: American call/put price r: risk-free rate

σ: share price volatility δ: continuous dividend yield I: fixed dividend at $t = 0$

Self-financing strategy

(ϕ_t, ψ_t) satisfying: $V(t) = \phi_t S_t + \psi_t B_t$ $dV(t) = \phi_t dS_t + \psi dB_t$

Replicating strategy

Self-financing (ϕ_t, ψ_t) satisfying: $\phi_T S_T + \psi_T B_T = X_T$ where X_T is the derivative payoff

Option payoff

European call: $\max(S_T - K, 0)$ European put: $\max(K - S_T, 0)$

Option bounds

Option Type	Upper bound	Lower bound
European call	$c_t \le S_t$	$c_t \ge S_t - K\exp[-r(T-t)]$
American call	$C_t \le S_t$	$C_t \ge S_t - K\exp[-r(T-t)]$
European put	$p_t \le K\exp[-r(T-t)]$	$p_t \ge K\exp[-r(T-t)] - S_t$
American put	$P_t \le K$	$P_t \ge K - S_t$

Parameter-price relationship

Increased parameter	Call price	Put price
Strike (K)	Decrease	Increase
Time to expiry $(T-t)$	Increase	Increase
Volatility (σ)	Increase	Increase
Risk-free rate (r)	Increase	Decrease
Dividend (δ)	Decrease	Increase
Share price (S_t)	Increase	Decrease

Forward price

zero dividend: $S_0 e^{rT}$ fixed dividend: $(S_0 - I)e^{rT}$ continuous dividend: $S_0 \exp[(r-\delta)T]$

One-period replicating portfolio

$$\Delta = \phi_0 = \frac{V_u - V_d}{S_0 e^{\delta h}(u-d)} \qquad B = \psi_0 = \frac{uV_d - dV_u}{e^{rh}(u-d)} \qquad V_0 = \Delta S_0 + B$$

Risk-neutral probability

$Q_{\text{(price increase)}} = q = \dfrac{e^{(r-\delta)h} - d}{u-d}$ $V_0 = e^{-rh}\mathrm{E}_Q[V(n)]$ Q: Risk-neutral Measure

Calibrating binomial model

$u = \exp\left[\sigma\sqrt{h} + \delta\,dt\right]$ $d = \exp\left[-\sigma\sqrt{h} + \delta\,dt\right]$ $h \to 0$: $\ln\dfrac{S_t}{S_0} \sim \mathrm{N}\left(\left(r - \dfrac{\sigma^2}{2}\right)t, \sigma^2 t\right)$

State-price deflator ($\delta = 0$)

$A_1 = e^{-r}\frac{q}{p}$ if $S_1 = S_0 u$ $\quad A_1 = e^{-r}\frac{1-q}{1-p}$ if $S_1 = S_0 d$ $\quad V_0 = \mathrm{E}_P[A_1 V_1]$

$A_n = e^{-rn}\left(\frac{q}{p}\right)^{N_n}\left(\frac{1-q}{1-p}\right)^{n-N_n}$

N_n: number of up's til time n $\quad V_0 = \mathrm{E}_P[A_n V_n]$

BLACK-SCHOLES MODEL

Equivalent measure $P \sim Q$ are equivalent $\iff$ $P(E) > 0$ whenever $Q(E) > 0$ for any event E.

Cameron-Martin-Girsanov There exists an equivalent measure Q s.t. $W_t + \int_0^t \gamma_s\,ds$ is a Wiener process, where γ_s is a previsible process.

Martingale discounted price To make $e^{-rt}S_t$ martingle, apply CMG with $\gamma_t = \frac{\mu - r}{\sigma}$, where $S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$

Martingale representation X_t, Y_t: martingale. Then, $dY_t = \phi_t\,dX_t$ for some ϕ_t $\iff$ X_t is not martingale under any other equivalent measure.

Black-Scholes formula ($\delta = 0$)

$c_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \qquad p_t = Ke^{-r(T-t)}N(-d_2) - S_t N(-d_1)$

$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$

Black-Scholes Equation ($\delta = 0$) $\frac{\partial f}{\partial t} + rS_t\frac{\partial f}{\partial s} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial s^2} = rf(t, S_t)$ $\quad f(T, s) = (s-K)_+$ for call, $(K-s)_+$ for put

BSE derivation ($\delta = 0$)

$V(t, S_t)$: portfolio value with 1 derivative and $\frac{\partial f}{\partial s}$ shares

$f = f(t, S_t)$: option value $\implies dV_t = -df + \frac{\partial f}{\partial s}\,dS_t$

Applying Itô's lemma:

$dV_t = \left(-\frac{\partial f}{\partial s} - \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 S_t^2\right)dt$

Arbitrage-free implies: $dV_t = rV_t\,dt$

Martingale approach ($\delta = 0$)

F_t: filtration, X: contingent derivative payment

Find equivalent measure Q s.t. $D_t = e^{-rt}S_t$ is martingale

$f(t, S_t) = \exp[-r(T-t)]\mathrm{E}_Q[X|F_t]$

Then, $E_t = e^{-rt}f(t, S_t)$ is martingale

For some ϕ_t: $dE_t = \phi_t\,dD_t$ (martingale representation)

Let $\psi_t = E_t - \phi_t D_t$

(ϕ_t, ψ_t) is self-financing.

Derivative value: $f(t, S_t) = \phi_t S_t + \psi_t B_t$

Martingale to BSE ($\delta = 0$)

$$dE_t = e^{-rt}(-rf(t, S_t)\,dt + df)$$
$$= \frac{\partial f}{\partial s}\,dD_t + e^{-rt}\left[-rf(t, S_t) + \frac{\partial f}{\partial t} + rS_t\frac{\partial f}{\partial s} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial s^2}\right]dt$$

Martingale representation: $dE_t = \phi_t\,dD_t \implies \phi_t = \frac{\partial f}{\partial s}$

Martingale $\implies -rf(t, S_t) + \frac{\partial f}{\partial t} + rS_t\frac{\partial f}{\partial s} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial s^2} = 0$

State-price deflator approach ($\delta = 0$)

$$dS_t \overset{P}{=} \mu S_t\,dt + \sigma S_t\,dZ_t$$
$$\overset{Q}{=} rS_t\,dt + \sigma S_t\,d\widetilde{Z}_t$$

$P \sim Q$: By CMG, $d\widetilde{Z}_t = dZ_t + \gamma_t\,dt$ with $\gamma_t = \frac{\mu - r}{\sigma}$

$\eta_t = \exp\left[-\gamma Z_t - \frac{1}{2}\gamma^2 t\right]$

State-price deflator: $A_t = e^{-rt}\eta_t$

$$f(t,S_t) = e^{-r(T-t)}\mathrm{E}_Q[X|F_t] = e^{-r(T-t)}\mathrm{E}_P\left[\frac{\eta_T}{\eta_t}X \,\middle|\, F_t\right] = \frac{\mathrm{E}_P[A_T X|F_t]}{A_t}$$

BSE derivation ($\delta > 0$)

$\widetilde{S}_t$: value of investment starting with S_0

$$d\widetilde{S}_t = (\mu+\delta)\widetilde{S}_t\,dt + \sigma\,dZ_t$$

$$\widetilde{S}_t = \widetilde{S}_0 \exp\left[\left(\mu+\delta-\frac{\sigma^2}{2}\right)t + \sigma Z_t\right]$$

Change in derivation:

$$dV_t = -df + \frac{\partial f}{\partial s}(dS_t + \delta S_t\,dt)$$

$$\frac{\partial f}{\partial t} + (r-\delta)S_t\frac{\partial f}{\partial s} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial s^2} = rf(t,S_t)$$

Black-Scholes formula ($\delta > 0$)

$$c_t = S_t e^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$p_t = Ke^{-r(T-t)}N(-d_2) - S_t e^{-\delta(T-t)}N(-d_1)$$

$$d_1 = \frac{\ln(S_t/K) + (r-\delta+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

Option Greeks

$$\Delta = \frac{\partial f}{\partial s},\ \Gamma = \frac{\partial^2 f}{\partial s^2},\ \Theta = \frac{\partial f}{\partial t},$$

$$\nu = \frac{\partial f}{\partial \sigma},\ \rho = \frac{\partial f}{\partial r},\ \lambda = \frac{\partial f}{\partial \delta}$$

BSE ($\delta = 0$):

$$\Theta + rs\Delta + \frac{1}{2}\sigma^2 s^2\Gamma = rf$$

$$\Delta_{\text{call}} = e^{-\delta(T-t)}N(d_1) \qquad \Delta_{\text{put}} = -e^{-\delta(T-t)}N(-d_1)$$