A. SEVERITY MODELS

A1. SEVERITY DISTRIBUTIONS

Distribution	Probability density function	Formulas worth memorizing		
Uniform	$f(x) = \frac{1}{b-a},\ a \le x \le b$	$F(x) = \frac{x-a}{b-a}$	$E[X] = \frac{a+b}{2}$	$Var(X) = \frac{(b-a)^2}{12}$
Exponential	$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}},\ x > 0$	$F(x) = 1 - e^{-\frac{x}{\theta}}$	$E[X] = \theta$	$Var(X) = \theta^2$
Weibull	$f(x) = \frac{\tau}{x}\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}},\ x > 0$	$F(x) = 1 - e^{-(\frac{x}{\theta})^{\tau}}$		
Gamma	$f(x) = \frac{\left(\frac{1}{\theta}\right)^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\theta}},\ x > 0$	$F(x) = \Pr(X^* \ge \alpha)$ X^* is Poisson with $\lambda = \frac{x}{\theta}$ If α is an integer.	$E[X] = \alpha\theta$	$Var(X) = \alpha\theta^2$
Beta	$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1},$ $0 < x < 1$		$E[X] = \frac{a}{a+b}$ If a and b are integers.	$E[X^2] = \frac{a(a+1)}{(a+b)(a+b+1)}$ If a and b are integers.
Pareto	$f(x) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}},\ x > 0$	$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}$	$E[X] = \frac{\theta}{\alpha-1}$ If a is an integer.	
Single P. Pareto	$f(x) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}},\ x > \theta$	$F(x) = 1 - \left(\frac{\theta}{x}\right)^{\alpha}$	$E[X] = \frac{\alpha\theta}{\alpha-1}$	
Lognormal	$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\log x-\mu)^2}{2\sigma^2}},\ x > 0$	$F(x) = N\left(\frac{\log x - \mu}{\sigma}\right)$	$E[X] = e^{\mu+\frac{\sigma^2}{2}}$	$E[X^2] = e^{2\mu+2\sigma^2}$

A2. SCALING & TRANSFORMATION

Scaling: $Y = cX \quad \rightarrow \quad F_Y(y) = F_X\left(\frac{y}{c}\right)$

Transformation: $Y = g(X) \quad \rightarrow \quad F_Y(y) = F_X\left(g^{-1}(y)\right)$ If g is monotonically increasing.

$\rightarrow \quad F_Y(y) = S_X\left(g^{-1}(y)\right)$ If g is monotonically decreasing.

A3. MIXING & SPLICING

Mixture: $f_X(x) = w_1 f_2(x) + w_2 f_2(x) + \cdots + w_k f_k(x)$ Where $w_1 + w_2 + \cdots + w_k = 1$.

Splices:

$$f_X(x) = \begin{cases} w_1 f_1(x), & x_0 \le x < x_1 \\ w_2 f_2(x), & x_1 \le x < x_2 \\ \dots & \\ w_k f_k(x), & x_{k-1} \le x < x_k \end{cases}$$

A4. TAILS & LIMITING DISTRIBUTIONS

Limit of ratio of two survival functions: $\lim_{x\to\infty} \frac{S_1(x)}{S_2(x)} = \lim_{x\to\infty} \frac{f_1(x)}{f_2(x)}$

Limit of hazard rate function: $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{f(x)}{S(x)}$

Limit of mean excess loss function: $\lim_{d\to\infty} e_X(d) = \lim_{d\to\infty} \frac{\int_d^\infty S(x)dx}{S(d)} = \lim_{d\to\infty} \frac{1}{h(d)}$

Equilibrium distribution: $f_e(x) = \frac{S(x)}{E[X]} \quad \rightarrow \quad E[X_e] = \frac{E[X^2]}{2E[X]}$

A5. PAYMENT PER LOSS

Policy	Payment per loss	Expected payment per loss
With ordinary deductible d	$Y^L = \begin{cases} 0, & X < d \\ X-d, & X \geq d \end{cases}$	$E[Y^L] = E[X] - E[X \wedge d]$
With franchise deductible d^*	$Y^L = \begin{cases} 0, & X \leq d^* \\ X, & X > d^* \end{cases}$	$E[Y^L] = E[X \mid X > d^*]$
With maximum covered loss u	$Y^L = \begin{cases} X, & X \leq u \\ u, & X > u \end{cases}$	$E[Y^L] = E[X \wedge u]$
With d and u	$Y^L = \begin{cases} 0, & X \leq d \\ X-d, & d < X \leq u \\ u-d, & X > u \end{cases}$	$E[Y^L] = E[X \wedge u] - E[X \wedge d]$
With d, u and coinsurance factor α	$Y^L = \begin{cases} 0, & X \leq d \\ \alpha(X-d), & d < X \leq u \\ \alpha(u-d), & X > u \end{cases}$	$E[Y^L] = \alpha(E[X \wedge u] - E[X \wedge d])$
With d, u, α and inflation rate r	$Y^L = \begin{cases} 0, & X \leq \frac{d}{1+r} \\ \alpha(1+r)\left(X - \frac{d}{1+r}\right), & \frac{d}{1+r} < X \leq \frac{u}{1+r} \\ \alpha(1+r)\left(\frac{u}{1+r} - \frac{d}{1+r}\right), & X > \frac{u}{1+r} \end{cases}$	$E[Y^L] = \alpha(1+r)\left(E\left[X \wedge \frac{u}{1+r}\right] - E\left[X \wedge \frac{d}{1+r}\right]\right)$

Loss elimination ratio: $\text{LER} = 1 - \frac{E[Y^L]}{E[X]}$

Increased limits factor: $\text{ILF}(u) = \frac{E[X \wedge u]}{E[X \wedge b]}$ Where b is the basic limit.

Indicated deductible relativity: $\text{IDR}(d) = \frac{E[X] - E[X \wedge d]}{E[X] - E[X \wedge b]}$ Where b is the basic deductible.

A6. PAYMENT PER PAYMENT

Payment per payment $Y^P = Y^L | X > d \quad \rightarrow \quad E[Y^P] = \frac{E[Y^L]}{\Pr(X > d)}$

Policy	Loss	Payment per payment
With ordinary deductible d	$X \sim Unif(0, b)$	$Y^P \sim Unif(0, b-d)$
	$X \sim Exp(\theta)$	$Y^P \sim Exp(\theta)$
	$X \sim Pareto(\alpha, \theta)$	$Y^P \sim Pareto(\alpha, \theta + d)$

A7. EXTREME VALUE DISTRIBUTIONS

Generalized Extreme Value Distribution (GEV):
$$H_\xi(x) = \begin{cases} \exp\left(-(1+\xi x)^{-\frac{1}{\xi}}\right), & \xi \neq 0,\ \xi x > -1, \\ \exp\left(-e^{-x}\right), & \xi = 0. \end{cases}$$

GEV with location and scale parameter:
$$H_{\xi,\mu,\theta}(x) = \begin{cases} \exp\left(-\left(1+\xi\left(\frac{x-\mu}{\theta}\right)\right)^{-1/\xi}\right) & \xi \neq 0,\ \xi\left(\frac{x-\mu}{\theta}\right) > -1 \\ \exp\left(-e^{-(x-\mu)/\theta}\right) & \xi = 0 \end{cases}$$

Fréchet distribution
$$F(x) = \exp\left(-\left(1+\xi\left(\frac{x-\mu}{\theta}\right)\right)^{-1/\xi}\right), \quad \xi > 0 \text{ and } x > \mu - \frac{\theta}{\xi}$$

Gumbel Distribution:
$$F(x) = \exp\left(-\exp\left(-\frac{x-\mu}{\theta}\right)\right)$$

Weibull EV distribution:
$$F(x) = \exp\left(-\left(1+\xi\left(\frac{x-\mu}{\theta}\right)\right)^{-1/\xi}\right), \quad \xi < 0 \text{ and } x < \mu - \frac{\theta}{\xi}$$

$F_X(x)$ is in the MDA of H_ξ if and only if $\lim_{n\to\infty} nS(c_n x + d_n) = -\ln H_\xi(x) = \begin{cases} (1+\xi x)^{-1/\xi} & \xi \neq 0 \\ e^{-x} & \xi = 0 \end{cases}$

Generalized Pareto Distribution (GPD):
$$G_{\xi,\beta}(x) = \begin{cases} 1-(1+\xi x/\beta)^{-1/\xi} & \xi \neq 0 \\ 1-e^{-x/\beta} & \xi = 0 \end{cases} \text{ where } \begin{array}{c} \beta > 0,\ x \geq 0 \text{ for } \xi \geq 0, \\ \text{and } 0 \leq x \leq -\beta/\xi \text{ for } \xi < 0 \end{array}$$

Relationship:
$$G_{\xi,\beta}(x) = 1 + \ln H_\xi(x/\beta)$$

Mean excess loss for GPD:
$$e_X(d) = \begin{cases} \frac{\beta+\xi d}{1-\xi}, & \text{for } 0 < \xi < 1 \\ \beta, & \text{for } \xi = 0 \\ \infty, & \text{for } \xi \geq 1 \end{cases}$$

Value-at-Risk (VaR):
$$Q_\alpha = d + \frac{\beta}{\xi}\left(\left(\frac{S_X(d)}{1-\alpha}\right)^\xi - 1\right)$$

Expected Shortfall:
$$\text{ES}_\alpha = Q_\alpha + e_X(Q_\alpha) = Q_\alpha + \frac{\beta+\xi(Q_\alpha - d)}{1-\xi} = \frac{Q_\alpha + \beta - \xi d}{1-\xi}$$

Hill estimator:
$$\hat{\alpha}_j^H = \left(\sum_{k=j}^{n} \frac{\ln x_{(k)}}{n-j+1} - \ln x_{(j)}\right)^{-1}$$

Hill estimate of survival function:
$$\hat{S}^H(x) = \frac{n-j}{n}\left(\frac{x}{x_{(j)}}\right)^{-\hat{\alpha}_j^H}$$

B. FREQUENCY MODELS

B1. FREQUENCY DISTRIBUTIONS

Distribution	Probability mass function	Formulas worth memorizing	
Poisson	$P(n) = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0,1,2,\ldots,\infty$	$E[N] = \lambda$	$Var(N) = \lambda$
Binomial	$P(n) = \binom{m}{n} q^n (1-q)^{m-n}, n = 0,1,2,\ldots,m$	$E[N] = mq$	$Var(N) = mq(1-q)$
Geometric	$P(n) = \left(\frac{1}{1+\beta}\right)\left(\frac{\beta}{1+\beta}\right)^n, n = 0,1,2,\ldots,\infty$	$E[N] = \beta$	$Var(N) = \beta(1+\beta)$
Negative Binomial	$P(n) = \binom{n+r-1}{n}\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)^n, n = 0,1,2,\ldots,\infty$	$E[N] = r\beta$	$Var(N) = r\beta(1+\beta)$

B2. MODIFIED DISTRIBUTIONS

The (a,b,0) class: $\Pr\left(N^L = n\right) = p_n$ Poisson, Binomial, Negative Binomial, and Geometric distributions.

The (a,b,1) class: $p_n^T = \frac{p_n}{1-p_0}$ **Zero-truncated distributions**.

$p_n^M = \left(1 - p_0^M\right)\left(\frac{p_n}{1-p_0}\right)$ **Zero-modified distributions**.

B3. NUMBER OF PAYMENTS

Probability of payment: $v = \Pr(X > d)$

Policy	**Number of losses**	**Number of payments**
With ordinary deductible d	$N^L \sim Poisson\,(\lambda)$	$N^P \sim Poisson\,(v\lambda)$
	$N^L \sim Binomial\,(m, q)$	$N^P \sim Binomial\,(m, vq)$
	$N^L \sim Geometric\,(\beta)$	$N^P \sim Geometric\,(v\beta)$
	$N^L \sim N.Binomial\,(r, \beta)$	$N^P \sim N.Binomial\,(r, v\beta)$

For modified distributions: $1 - p_0^{M*} = \left(1 - p_0^M\right)\left(\frac{1-p_0^*}{1-p_0}\right)$

Asterisks indicate distributions with revised parameters.

C. AGGREGATE MODELS

C1. AGGREGATE LOSSES

Aggregate losses: $S^L = X_1 + X_2 + \cdots + X_{N^L}$

Mean: $E\left[S^L\right] = E\left[N^L\right] E[X]$

Variance: $\operatorname{Var}\left(S^L\right) = E\left[N^L\right] \operatorname{Var}(X) + \operatorname{Var}\left(N^L\right) E[X]^2$

Probability: $\Pr\left(S^L \le k\right) = \sum_{n=0}^{\infty} \Pr\left(N^L = n\right) \Pr\left(X_1 + \cdots + X_n \le k\right)$

C2. AGGREGATE PAYMENTS

Aggregate Payments: $S^P = Y_1^L + Y_2^L + \cdots + Y_{N^L}^L$ $\quad S^P = Y_1^P + Y_2^P + \cdots + Y_{N^P}^P$

Mean: $E\left[S^P\right] = E\left[N^L\right] E\left[Y^L\right]$ $\quad E\left[S^P\right] = E\left[N^P\right] E\left[Y^P\right]$

Variance: $\operatorname{Var}\left(S^L\right) = E\left[N^L\right] \operatorname{Var}\left(Y^L\right) + \operatorname{Var}\left(N^L\right) E\left[Y^L\right]^2$

$\operatorname{Var}\left(S^P\right) = E\left[N^P\right] \operatorname{Var}\left(Y^P\right) + \operatorname{Var}\left(N^P\right) E\left[Y^P\right]^2$

Probability:

$$\Pr\left(S^P \le k\right) = \sum_{n=0}^{\infty} \Pr\left(N^L = n\right) \Pr\left(Y_1^L + \cdots + Y_n^L \le k\right)$$

$$\Pr\left(S^P \le k\right) = \sum_{n=0}^{\infty} \Pr\left(N^P = n\right) \Pr\left(Y_1^P + \cdots + Y_n^P \le k\right)$$

C3. AGGREGATE DEDUCTIBLE

Aggregate deductible: d^s

Aggregate payments:

$$S^P = \begin{cases} 0, & S^L < d^s \\ S^L - d^s, & S^L \ge d^s \end{cases} \rightarrow \quad E\left[S^P\right] = E\left[S^L\right] - E\left[S^L \wedge d^S\right]$$

C4. DISCRETIZING

Method of rounding:

$$f_0 = \Pr\left(X < \tfrac{h}{2}\right) = F_X\left(\tfrac{h}{2} - 0\right)$$

$$f_j = \Pr\left(jh - \tfrac{h}{2} \le X < jh + \tfrac{h}{2}\right) = F_X\left(jh + \tfrac{h}{2} - 0\right) - F_X\left(jh - \tfrac{h}{2} - 0\right) \text{ for } j = 1, 2, \ldots$$

Method of Mean Preserving:

$$f_0 = 1 - \frac{E[X \wedge h]}{h}$$

$$f_i = \frac{2E[X \wedge ih] - E[X \wedge (i-1)h] - E[X \wedge (i+1)h]}{h} \text{ for } j = 1, 2, \ldots$$

D. MAXIMUM LIKELIHOOD ESTIMATION

D1. MLE WITH COMPLETE DATA

For distributions that belong to the exponential family:

1. Determine $L(\theta)$.
2. Apply natural logarithm, obtain $l(\theta) = \log L(\theta)$.
3. Take the first derivative with respect to the parameter, obtain $l'(\theta)$.
4. Set $l'(\theta) = 0$, obtain $\hat{\theta}$, which is the MLE.

Distribution	Likelihood Function	Maximum likelihood estimate(*s*)	
Exponential	$L(\theta) = f(x_1)\ldots f(x_n)$	$\hat{\theta} = \bar{x}$	
Gamma		$\hat{\theta} = \frac{\bar{x}}{\alpha}$	
Normal		$\hat{\mu} = \bar{x}$	$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \hat{\mu})^2$
Lognormal		$\hat{\mu} = \frac{1}{n}\sum \log x_i$	$\hat{\sigma}^2 = \frac{1}{n}\sum(\log x_i - \hat{\mu})^2$
Uniform		$\widehat{b} = \max(x_1, \ldots, x_n)$	
Binomial	$L(\theta) = p(x_1)\ldots p(x_n)$	$\hat{q} = \frac{\bar{x}}{m}$	
Poisson		$\hat{\lambda} = \bar{x}$	
Negative Binomial		$\hat{\beta} = \frac{\bar{x}}{r}$	

D2. MLE WITH INCOMPLETE DATA

	Likelihood function	Note
Grouped data	$L = (F(c_1) - F(c_0))^{n_1} \dots (F(c_j) - F(c_{j-1}))^{n_j}$	Where $c_0 < c_1 < \cdots < c_n$ are interval boundaries.
Left-truncated data	$L = \frac{f(x_1)}{S(d)} \dots \frac{f(x_n)}{S(d)}$	Losses below d are not reported.
Right-censored data	$L = f(x_1) \dots f(x_n) S(u)^m$	Losses are capped at u.
Left-truncated & Right-censored data	$L = \frac{f(x_1)}{S(d)} \dots \frac{f(x_n)}{S(d)} \left(\frac{S(u)}{S(d)}\right)^m$	Losses below d are not reported. Losses are capped at u.

D3. VARIANCE OF MLE

Number of parameters	Information matrix	Variance of MLE
1	$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} l(\theta)\right]$	$\text{Var}(\hat{\theta}) = I(\theta)^{-1}$
2	$I(\theta_1, \theta_2) = -E\begin{bmatrix} \frac{\partial^2}{\partial\theta_1^2} l(\theta_1, \theta_2) & \frac{\partial^2}{\partial\theta_1\partial\theta_2} l(\theta_1, \theta_2) \\ \frac{\partial^2}{\partial\theta_1\partial\theta_2} l(\theta_1, \theta_2) & \frac{\partial^2}{\partial\theta_2^2} l(\theta_1, \theta_2) \end{bmatrix}$	$VCOV\left(\hat{\theta}_1, \hat{\theta}_2\right) = I(\theta_1, \theta_2)^{-1} = \begin{bmatrix} \text{Var}\left(\hat{\theta}_1\right) & \text{Cov}\left(\hat{\theta}_1, \hat{\theta}_2\right) \\ \text{Cov}\left(\hat{\theta}_1, \hat{\theta}_2\right) & \text{Var}\left(\hat{\theta}_2\right) \end{bmatrix}$

Inverting a matrix: $$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Normal CI for θ: $\hat{\theta} \pm z\sqrt{\widehat{\text{Var}}(\hat{\theta})}$ Where $\widehat{\text{Var}}(\hat{\theta})$ is an estimate of $\text{Var}(\hat{\theta})$.

D4. DELTA METHOD

Number of parameters	Function of MLE	Variance using Delta method
1	g is a function of θ $\hat{g}$ is an estimator of g, using $\hat{\theta}$.	$\text{Var}(\hat{g}) \approx \text{Var}(\hat{\theta}) \left(\frac{\partial g}{\partial \theta}\right)^2$
2	h is a function of θ_1 and θ_2. $\hat{h}$ is an estimator of h, using $\hat{\theta}_1$ and $\hat{\theta}_2$.	$\text{Var}(\hat{h}) \approx \text{Var}\left(\hat{\theta}_1\right)\left(\frac{\partial h}{\partial\theta_1}\right)^2 + \text{Var}\left(\hat{\theta}_2\right)\left(\frac{\partial h}{\partial\theta_2}\right)^2 + 2\,\text{Cov}\left(\hat{\theta}_1, \hat{\theta}_2\right)\left(\frac{\partial h}{\partial\theta_1}\right)\left(\frac{\partial h}{\partial\theta_2}\right)$

E. MODEL SELECTION

E1. D(X) PLOTS & P-P PLOTS

D(x) plots: x on the x axis. $D(x) = F_n(x) - F^*(x)$ on the y axis.

$F_n(x)$ is the **empirical distribution function**.

$F^*(x)$ is the **fitted distribution function.**

p-p plots: $F_n(x)$ on the x axis. $F^*(x)$ on the y axis.

E2. KOLMOGOROV-SMIRNOV TEST

Null hypothesis: The parametric model fits its data well

Test statistic: $D = \max D_j$ where $D_j = \max\left(\left|F^*(x_j) - \frac{j}{n}\right|, \left|F^*(x_j) - \frac{j+1}{n}\right|\right)$ F^* is the fitted CDF.

1-tailed test. Reject H_0 if $D > c$.

E3. CHI-SQUARE TEST

Null hypothesis: The parametric model fits its data well / The mean is the same across categories

Suppose there are k categories:

O_i is the observed value for category i.

Test statistic: $Q = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(k-1)$ E_i is the expected value for category i.

1-tailed test. Reject H_0 if $Q > c$.

Suppose there are $k_1 \times k_2$ categories:

Test statistic: $Q = \sum_{j=1}^{k_2} \sum_{i=1}^{k_1} \frac{(o_{ij} - E_{ij})^2}{E_{ij}} \sim \chi^2\left((k_1 - 1)(k_2 - 1)\right)$

1-tailed test. Reject H_0 if $Q > c$.

Note: Subtract additional 1 degree of freedom for each parameter fitted from the data.

E4. LIKELIHOOD RATIO TEST

$\boldsymbol{H_0 : \theta = \theta_0}$ **vs** $\boldsymbol{H_1 : \theta \neq \theta_0}$

$R = \frac{L(X|\theta_0)}{L(X|\theta_1)}$ Where θ_1 can be the maximum likelihood estimate of θ.

Test Statistic $-2\log R \sim \chi^2(1)$ The DOF depends on the number of parameters specified in H_0 and H_1.

1-tailed test. Reject H_0 if test statistic $> c$.

E5. SBC & AIC

Schwartz Bayesian Criterion: $\log L - \frac{r}{2}\log n$ — r is the number of parameters in the model.

Akaike Information Criterion: $\log L - r$

Note: The model with higher SBC/AIC is better.

F. CREDIBILITY

F1. BAYESIAN CREDIBILITY

Observations: $\vec{X} = \{X_1 = x_1, \ldots, X_n = x_n\}$

Prior pmf: $\Pr(\theta = k)$ if θ is discrete.

Prior pdf: $f(\theta = k)$ if θ is continuous.

Posterior pmf/pdf

	$\boldsymbol{X}$ is discrete	$\boldsymbol{X}$ is continuous
$\boldsymbol{\theta}$ is discrete	$\Pr(\theta = k \mid \vec{\boldsymbol{X}}) = \frac{\Pr(\theta=k)\Pr(\vec{\boldsymbol{X}}\mid\theta=k)}{\Pr(\vec{\boldsymbol{X}})}$	$\Pr(\theta = k \mid \vec{\boldsymbol{X}}) = \frac{\Pr(\theta=k)f(\vec{\boldsymbol{X}}\mid\theta=k)}{f(\vec{\boldsymbol{X}})}$
$\boldsymbol{\theta}$ is continuous	$f(\theta = k \mid \vec{\boldsymbol{X}}) = \frac{f(\theta=k)\Pr(\vec{\boldsymbol{X}}\mid\theta=k)}{\Pr(\vec{\boldsymbol{X}})}$	$f(\theta = k \mid \vec{\boldsymbol{X}}) = \frac{f(\theta=k)f(\vec{\boldsymbol{X}}\mid\theta=k)}{f(\vec{\boldsymbol{X}})}$

Predictive Probability

$\Pr\left(X_{n+1} \le x \mid \vec{\boldsymbol{X}}\right) = \sum \Pr(X \le x \mid \theta = k)\Pr(\theta = k \mid \vec{\boldsymbol{X}})$ If θ is discrete.

$\Pr\left(X_{n+1} \le x \mid \vec{\boldsymbol{X}}\right) = \int \Pr(X \le x \mid \theta = k) f(\theta = k \mid \vec{\boldsymbol{X}})dk$ If θ is continuous.

Bayesian Premium

$E\left[X_{n+1} \mid \vec{\boldsymbol{X}}\right] = \sum E[X \mid \theta = k]\Pr(\theta = k \mid \vec{\boldsymbol{X}})$ If θ is discrete.

$E\left[X_{n+1} \mid \vec{\boldsymbol{X}}\right] = \int E[X \mid \theta = k] f(\theta = k \mid \vec{\boldsymbol{X}})dk$ If θ is continuous.

Distribution	Conjugate prior	Posterior
$X \mid \lambda \sim \text{Poisson}(\lambda)$	$\lambda \sim \text{Gamma}(\alpha, \theta)$	$\lambda \mid \vec{\boldsymbol{X}} \sim \text{Gamma}\left(\alpha^* = \alpha + \sum X_i, \theta^* = \frac{1}{\frac{1}{\theta}+n}\right)$
$X \mid \mu \sim \text{Normal}\left(\mu, \sigma^2\right)$	$\mu \sim \text{Normal}(m, v)$	$\mu \mid \vec{\boldsymbol{X}} \sim \text{Normal}\left(m^* = \frac{\sigma^2 m + nv\bar{X}}{\sigma^2+nv}, v^* = \frac{\sigma^2 v}{\sigma^2+nv}\right)$
$X \mid q \sim \text{Bernoulli}\ (q)$	$q \sim \text{Beta}(a, b)$	$q \mid \vec{\boldsymbol{X}} \sim \text{Beta}\left(a^* = a + \sum X_i, b^* = b + n - \sum X_i\right)$
$X \mid Y \sim \text{Exponential}\ \left(\theta = \frac{1}{Y}\right)$	$Y \sim \text{Gamma}(\alpha, \theta)$	$Y \mid \vec{\boldsymbol{X}} \sim \text{Gamma}\left(\alpha^* = \alpha + n, \theta^* = \frac{1}{\frac{1}{\theta}+\sum x_i}\right)$

F2. BÜHLMANN CREDIBILITY

Expected Value of Hypothetical Mean: $\mu = E\left[E\left[X|\theta\right]\right]$

Expected Value of Process Variance: $v = E\left[Var\left(X|\theta\right)\right]$

Variance of Hypothetical Mean: $a = Var\left(E\left[X|\theta\right]\right)$

Credibility Factor: $Z = \dfrac{n}{n + \dfrac{v}{a}}$

Bühlmann Premium: $E\left[X_{n+1}\middle|\vec{X}\right] = Z\bar{X} + (1-Z)\,\mu$

Note: The Bühlmann estimate is a linear approximation of the Bayesian estimate. If the Bayesian estimate is a linear function of the sample mean, then the Bühlmann estimate is equal to the Bayesian estimate.

F3. NON-PARAMETRIC EMPIRICAL BAYES CREDIBILITY

Equal Sample Size:

$$\bar{X}_i = \frac{\sum_{j=1}^{n} X_{ij}}{n}$$

$$\hat{\mu} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n} X_{ij}}{rn} \qquad \hat{v} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n}\left(X_{ij}-\bar{X}_i\right)^2}{r\,(n-1)} \qquad \hat{a} = \frac{\sum_{i=1}^{r}\left(\bar{X}_i-\hat{\mu}\right)^2}{r-1} - \frac{\hat{v}}{n}$$

Credibility factor: $Z = \dfrac{n}{n + \dfrac{\hat{v}}{\hat{a}}}$

Credibility premium: $Z\bar{X}_i + (1-Z)\,\hat{\mu}$

Unequal Sample Size:

$$\bar{X}_i = \frac{\sum_{j=1}^{n_i} m_{ij}X_{ij}}{m_i}$$

$$\hat{\mu} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij}X_{ij}}{m} \qquad \hat{v} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij}\left(X_{ij}-\bar{X}_i\right)^2}{\sum_{i=1}^{r}(n_i-1)} \qquad \hat{a} = \frac{\sum_{i=1}^{r} m_i\left(\bar{X}_i-\hat{\mu}\right)^2 - \hat{v}\,(r-1)}{m - \frac{1}{m}\sum_{i=1}^{r} m_i^2}$$

Credibility factor: $Z = \dfrac{m_i}{m_i + \dfrac{\hat{v}}{\hat{a}}}$

Credibility premium: $Z\bar{X}_i + (1-Z)\,\hat{\mu}$

F4. SEMI-PARAMETRIC EMPIRICAL BAYES CREDIBILITY

Poisson Model: $\hat{\mu} = \bar{X}$ $\qquad \hat{v} = \bar{X}$ $\qquad \hat{a} = S^2 - \hat{v}$

Credibility factor: $Z = \dfrac{n}{n + \dfrac{\hat{v}}{\hat{a}}}$

Credibility premium: $Z\bar{X}_i + (1-Z)\,\hat{\mu}$

G. RATEMAKING & LOSS RESERVING

G1. LOSS RESERVING METHODS

Expected Loss Ratio Method	1. Ultimate Losses = Earned Premium × Expected Loss Ratio 2. Loss Reserve = Ultimate Losses − Paid Losses
Chain-ladder Method	1. Prepare a run-off triangle for paid losses. 2. Calculate age-to-age factors using average factor method or mean factor method. 3. Calculate age-to-ultimate factor f_{ULT}, which is the product of age-to-age factors. 4. Ultimate Losses = Paid Losses × f_{ULT} 5. Loss reserve = Ultimate Losses − Paid Losses
Bornhuetter -Ferguson method	1. Prepare a run-off triangle for paid losses. 2. Calculate age-to-age factors using average factor method or mean factor method. 3. Calculate age-to-ultimate factor f_{ULT}, which is the product of age-to-age factors. 4. Ultimate Losses = Paid Losses + Earned Premium × Expected Loss Ratio × $\left(1 - \frac{1}{f_{ULT}}\right)$ 5. Loss reserve = Ultimate Losses − Paid Losses
Projecting Severity and Frequency Separately	1. Prepare a run-off triangle for severity and estimate ultimate severity using the chain-ladder method. 2. Prepare a run-off triangle for frequency and estimate ultimate frequency using the chain-ladder method. 3. Ultimate losses = Ultimate Severity × Ultimate Frequency 4. Loss reserve = Ultimate Losses − Paid Losses
Closure method	1. Prepare a run-off triangle for incremental payments and incremental claim counts. 2. Divide incremental payments by incremental claim counts to obtain incremental severity. 3. Trend incremental severity, average, and detrend. 4. Calculate annual claim closure percentages, average, and project claim counts. 5. Loss reserve = Projected Incremental Severity × Projected Incremental Closed Claims
Discounted Loss Reserves	1. Project incremental payments using one of the above methods. 2. Discounted loss reserve = PV of projected incremental payments

G2. PRICING FORMULA

General formula: Premium $=$ Losses $+$ Loss adjustment expenses $+$ Fixed Expenses $+$ Variable Expenses $+$ Profit

Premium: $P = L + LAE + F + (V + Q)P \quad \rightarrow \quad P = \dfrac{L + LAE + F}{1 - V - Q}$

Permissible loss ratio: $R = 1 - V - Q \quad \rightarrow \quad P = \dfrac{L + LAE + F}{R}$

Adjustments to data:

$$\text{Premium at current rates} = \text{Earned premium} \times \frac{\text{Current rate level}}{\text{Historical average rate level}}$$

$$\text{Ultimate losses} = \text{Reported losses} \times \text{Development factor}$$

$$\text{Trended losses} = \text{Reported losses} \times \text{Trend factor}$$

G3. OVERALL RATE INDICATION

Loss cost method or **Pure premium method**	$\text{Projected loss cost including LAE} = \dfrac{\text{Trended and ultimate losses and LAE}}{\text{Number of earned exposures}}$ $\text{Indicated rate} = \dfrac{\text{Projected loss cost} + \text{Fixed expenses per exposure}}{\text{Permissible loss ratio}}$ $\text{Indicated rate change} = \dfrac{\text{Indicated rate}}{\text{Current rate}} - 1$
Loss ratio method	$\text{Projected loss and LAE ratio} = \dfrac{\text{Trended and ultimate losses and LAE}}{\text{Earned premiums at current rate level}}$ $\text{Indicated rate change} = \dfrac{\text{Projected loss and LAE ratio} + \text{Fixed expense ratio}}{\text{Permissible loss ratio}} - 1$ $\text{Indicated rate} = \dfrac{\text{Earned premiums at current rate level}}{\text{Number of earned exposures}} \times (1 + \text{Indicated rate change})$

G4. RISK CLASSIFICATION

Loss cost method: $\text{Indicated differential}_i = \dfrac{\text{Loss cost}_i}{\text{Loss cost}_{\text{base}}}$

Loss ratio method: $\text{Indicated differential}_i = \text{Existing differential}_i \times \dfrac{\text{Loss ratio}_i}{\text{Loss ratio}_{\text{base}}}$

Balancing back:

$$\text{Indicated overall rate} = \text{Current base rate} \times (1 + \text{Indicated overall rate change})$$

$$\text{Balance Back Factor} = \frac{\text{Average existing differential}}{\text{Average indicated differential}}$$

$$\text{Indicated rate}_i = \text{Indicated overall rate} \times \text{Balance Back Factor} \times \text{Indicated differential}_i$$