

Part A: MULTIPLE STATE MODELS

A1. MARKOV CHAIN PROBABILITIES

Transition probability:

${}_t p_x^{ij}$ is the probability that a life aged x who is currently in state i will be in state j at time t .

${}_t p_x^{ii}$ is the probability that a life aged x who is currently in state i will be in state i at time t .

${}_t \bar{p}_x^{ii}$ is the probability that a life aged x who is currently in state i will be in state i until time t .

$${}_t \bar{p}_x^{ii} \leq {}_t p_x^{ii}$$

$${}_0 p_x^{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Chapman-Kolmogorov equations:

$${}_k p_x^{ij} = \sum_s {}_r p_x^{is} {}_{k-r} p_{x+r}^{sj}$$

Forces of transition:

$$\mu_{x+t}^{ij} \rightarrow {}_t \bar{p}_x^{ii} = e^{-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij} ds}$$

Kolmogorov's forward equations:

$$\frac{d}{dt} {}_t p_x^{ij} = \frac{{}_{t+h} p_x^{ij} - {}_t p_x^{ij}}{h} = \sum_{k \neq j} \left({}_t p_x^{ik} \mu_{x+t}^{kj} - {}_t p_x^{ij} \mu_{x+t}^{jk} \right)$$

A2. ESTIMATION OF TRANSITION INTENSITIES

Define the following: $T^{(i)}$ is the total waiting time in State i between ages x and $x + 1$

D^{ij} is the number of direct transitions from State i to State j between ages x and $x + 1$

MLE of μ_x^{ij} :

$$\hat{\mu}_x^{ij} = \frac{D^{ij}}{T^{(i)}}$$

Variance:

$$\text{Var}(\hat{\mu}_x^{ij}) \approx \frac{D^{ij}}{(T^{(i)})^2}$$

A3. INSURANCE & ANNUITIES

Insurance functions:

$$A_x^{ij} = \sum_{k=0}^{\infty} v^{k+1} ({}_{k+1} p_x^{ij} - {}_k p_x^{ij})$$

$$\bar{A}_x^{ij} = \int_0^{\infty} v^t \sum_{k \neq j} {}_t p_x^{ik} \mu_{x+t}^{kj} dt$$

Annuity functions:

$$\ddot{a}_x^{ij} = \sum_{k=0}^{\infty} v^k {}_k p_x^{ij}$$

$$\bar{a}_x^{ij} = \int_0^{\infty} v^t {}_t p_x^{ij} dt$$

Relations:

$$A_x^{ij} = A_{x:\overline{n}|}^{ij} + \sum_k v^n {}_n p_x^{ik} A_{x+n}^{kj}$$

$$\bar{A}_x^{ij} = \bar{A}_{x:\overline{n}|}^{ij} + \sum_k v^n {}_n p_x^{ik} \bar{A}_{x+n}^{kj}$$

$$\ddot{a}_x^{ij} = \ddot{a}_{x:\overline{n}|}^{ij} + \sum_k v^n {}_n p_x^{ik} \ddot{a}_{x+n}^{kj}$$

$$\bar{a}_x^{ij} = \bar{a}_{x:\overline{n}|}^{ij} + \sum_k v^n {}_n p_x^{ik} \bar{a}_{x+n}^{kj}$$

Continuous sojourn annuity: $\bar{a}_{x:\overline{n}|}^{ii} = \int_0^n v^t {}_t p_x^{ii} dt$

Woolhouse formula, 2 terms: $\ddot{a}_x^{(m)ij} \approx \bar{a}_x^{ij} \qquad \ddot{a}_x^{(m)ii} \approx \bar{a}_x^{ii} + \frac{1}{2m}$

Woolhouse formula, 3 terms: $\ddot{a}_x^{(m)ij} \approx \bar{a}_x^{ij} - \frac{1}{12m^2} \mu_x^{ij}$ $\ddot{a}_x^{(m)ii} \approx \bar{a}_x^{ii} + \frac{1}{2m} + \frac{1}{12m^2} \left(\delta + \sum_{j \neq i} \mu_x^{ij} \right)$
 $\ddot{a}_x^{(m)ij} \approx \ddot{a}_x^{ij} + \frac{m^2-1}{12m^2} \mu_x^{ij}$ $\ddot{a}_x^{(m)ii} = \ddot{a}_x^{ii} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2} \left(\delta + \sum_{j \neq i} \mu_x^{ij} \right)$

A4. PREMIUMS & RESERVES

Let: ${}_t V^{(i)}$ be the reserve at time t if the life is in state i at that time.

$I^{(i)}$ be the premium (calculated using the equivalence principle) less benefit while the life is in state i .

$b^{(ij)}$ be the (usually one-time) benefit when the life transitions from state i to state j .

Recursive formula:

$$({}_k V^{(i)} + I^{(i)})(1+i) = \sum_j (b^{(ij)} + {}_{k+1} V^{(j)}) p_{x+k}^{ij}$$

$$({}_k V^{(i)} + sI^{(i)})(1+i)^s = \sum_j (b^{(ij)} + {}_{k+s} V^{(j)}) s p_{x+k}^{ij} \quad \text{Where } 0 < s < 1.$$

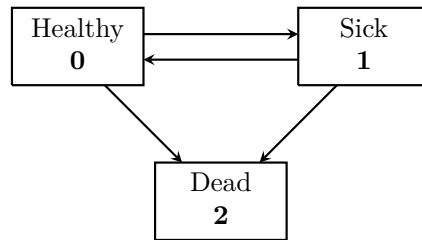
Thiele's differential equations:

$$\frac{d}{dt} {}_t V^{(i)} = \frac{{}_{t+h} V^{(i)} - {}_t V^{(i)}}{h} = {}_t V^{(i)} \left(\delta_t + \sum_{j \neq i} \mu_{x+t}^{ij} \right) + I_t^{(i)} - \sum_{j \neq i} (b_t^{(ij)} + {}_t V^{(j)}) \mu_{x+t}^{ij} \quad \text{Euler's forward method}$$

$$\frac{d}{dt} {}_t V^{(i)} = \frac{{}_t V^{(i)} - {}_{t-h} V^{(i)}}{h} = {}_t V^{(i)} \left(\delta_t + \sum_{j \neq i} \mu_{x+t}^{ij} \right) + I_t^{(i)} - \sum_{j \neq i} (b_t^{(ij)} + {}_t V^{(j)}) \mu_{x+t}^{ij} \quad \text{Euler's backward method}$$

A5. DISABILITY INCOME INSURANCE MODEL

Discrete DII model:



Transition matrix:

$$\mathbf{P}^{(t)} = \begin{pmatrix} p_{x+t}^{00} & p_{x+t}^{01} & p_{x+t}^{02} \\ p_{x+t}^{10} & p_{x+t}^{11} & p_{x+t}^{12} \\ p_{x+t}^{20} & p_{x+t}^{21} & p_{x+t}^{22} \end{pmatrix} = \begin{pmatrix} p_{x+t}^{00} & p_{x+t}^{01} & p_{x+t}^{02} \\ p_{x+t}^{10} & p_{x+t}^{11} & p_{x+t}^{12} \\ 0 & 0 & 1 \end{pmatrix}$$

Transition Probabilities: ${}_2p_x^{00} = p_x^{00}p_{x+1}^{00} + p_x^{01}p_{x+1}^{10} + p_x^{02}p_{x+1}^{20}$

$${}_2\bar{p}_x^{00} = p_x^{00}p_{x+1}^{00}$$

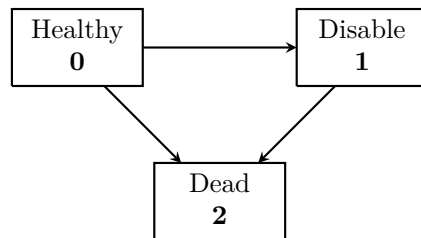
$${}_2p_x^{10} = \begin{pmatrix} p_x^{10} & p_x^{11} & p_x^{12} \end{pmatrix} \begin{pmatrix} p_{x+1}^{00} \\ p_{x+1}^{10} \\ 0 \end{pmatrix}$$

$${}_3p_x^{10} = \begin{pmatrix} p_x^{10} & p_x^{11} & p_x^{12} \end{pmatrix} \begin{pmatrix} p_{x+1}^{00} & p_{x+1}^{01} & p_{x+1}^{02} \\ p_{x+1}^{10} & p_{x+1}^{11} & p_{x+1}^{12} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{x+2}^{00} \\ p_{x+2}^{10} \\ 0 \end{pmatrix}$$

$${}_4p_x^{10} = \begin{pmatrix} p_x^{10} & p_x^{11} & p_x^{12} \end{pmatrix} \begin{pmatrix} p_{x+1}^{00} & p_{x+1}^{01} & p_{x+1}^{02} \\ p_{x+1}^{10} & p_{x+1}^{11} & p_{x+1}^{12} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{x+2}^{00} & p_{x+2}^{01} & p_{x+2}^{02} \\ p_{x+2}^{10} & p_{x+2}^{11} & p_{x+2}^{12} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{x+3}^{00} \\ p_{x+3}^{10} \\ 0 \end{pmatrix}$$

A6. PERMANENT DISABILITY MODEL

Continuous PD model:



Transition Intensities:

$$\mu_{x+t}^{01}$$

$$\mu_{x+t}^{02}$$

$$\mu_{x+t}^{12}$$

Transition Probabilities:

$${}_t p_x^{00} = e^{-\int_0^t (\mu_{x+s}^{01} + \mu_{x+s}^{02}) ds}$$

$${}_t p_x^{01} = \int_0^t {}_s p_x^{00} \mu_{x+s}^{01} {}_{t-s} p_x^{11} ds$$

$${}_t p_x^{02} = 1 - {}_t p_x^{00} - {}_t p_x^{01}$$

$${}_t p_x^{10} = 0$$

$${}_t p_x^{11} = e^{-\int_0^t \mu_{x+s}^{12} ds}$$

$${}_t p_x^{12} = 1 - {}_t p_x^{11}$$

$${}_t p_x^{20} = 0$$

$${}_t p_x^{21} = 0$$

$${}_t p_x^{22} = 1$$

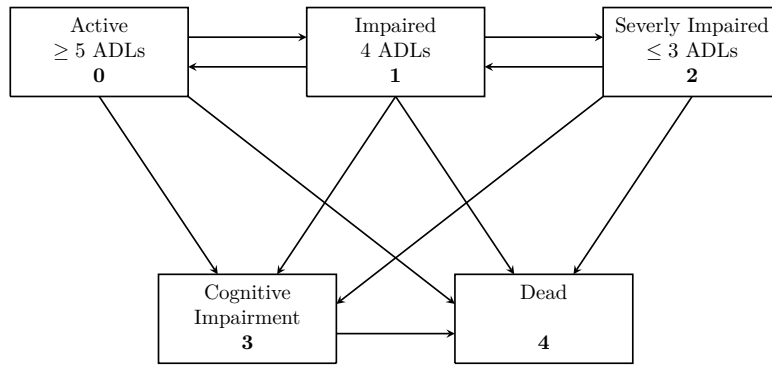
EPV of a benefit of 1 paid continuously while sick to a life currently aged x and healthy, with a waiting period of w years, and a policy term of $n > w$ years:

$$\bar{a}_{x:\overline{n}|}^{01} = \int_0^n v^t {}_t p_x^{00} \mu_{x+t}^{01} \bar{a}_{x+t:\overline{n-t}|}^{11} dt \quad \text{If } w = 0.$$

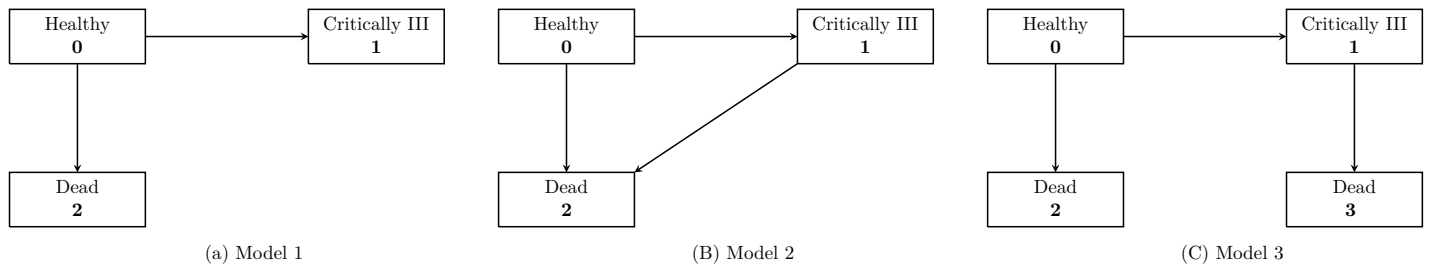
$$EPV = \int_0^{n-w} v^t {}_t p_x^{00} \mu_{x+t}^{01} \left(\bar{a}_{x+t:\overline{n-t}|}^{11} - \bar{a}_{x+t:\overline{w}|}^{11} \right) dt \quad \text{If } w > 0.$$

A7. OTHER MARKOV CHAIN MODELS

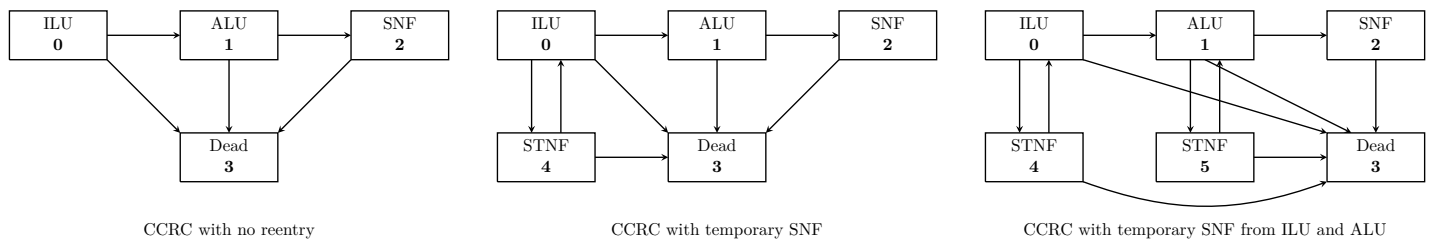
Long Term Care Insurance:



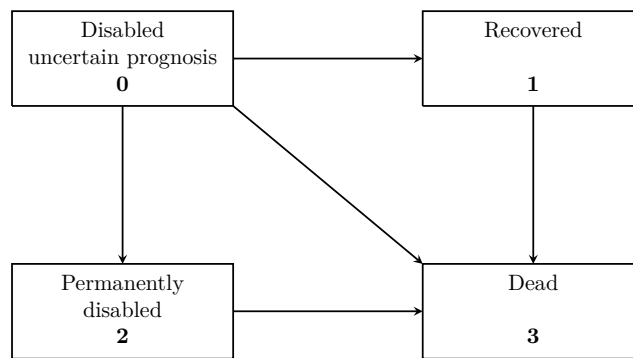
Critical Illness and Chronic Illness Insurance:



Continuing Care Retirement Communities:



Structured Settlements:



Reviewable annuity model

Part B: MULTIPLE DECREMENT MODELS

B1. MULTIPLE DECREMENT PROBABILITIES

Forces of decrement: $\mu_{x+s}^{(\tau)} = \sum_j \mu_{x+s}^{(j)} \rightarrow {}_t p_x^{(\tau)} = e^{-\int_0^t \mu_{x+s}^{(\tau)} ds}$

$\rightarrow {}_t q_x^{(\tau)} = \int_0^t {}_s p_x^{(\tau)} \mu_{x+s}^{(\tau)} ds \rightarrow \mu_{x+t}^{(\tau)} = \frac{\frac{d}{dt} {}_t q_x^{(\tau)}}{{}_t p_x^{(\tau)}}$

Multiple decrement probabilities: ${}_t p_x^{(j)}$ does not exist.

${}_t q_x^{(j)} = \int_0^t {}_s p_x^{(\tau)} \mu_{x+s}^{(j)} ds \rightarrow {}_t q_x^{(\tau)} = \sum_j {}_t q_x^{(j)} \rightarrow \mu_{x+t}^{(j)} = \frac{\frac{d}{dt} {}_t q_x^{(j)}}{{}_t p_x^{(\tau)}}$

Associated single decrement probabilities:

${}_t p_x'^{(j)} = e^{-\int_0^t \mu_{x+s}^{(j)} ds} \rightarrow {}_t p_x^{(\tau)} = \prod_j {}_t p_x^{(j)}$

${}_t q_x'^{(j)} = \int_0^t {}_s p_x'^{(j)} \mu_{x+s}^{(j)} ds \rightarrow \mu_{x+t}^{(j)} = \frac{\frac{d}{dt} {}_t q_x'^{(j)}}{{}_t p_x'^{(j)}}$

Formulas:

${}_{t+u} p_x^{(\tau)} = {}_t p_x^{(\tau)} {}_u p_{x+t}^{(\tau)} \quad {}_{t+u} q_x^{(\tau)} = {}_t q_x^{(\tau)} + {}_t p_x^{(\tau)} {}_u q_{x+t}^{(\tau)}$

${}_{t+u} q_x^{(j)} = {}_t q_x^{(j)} + {}_t p_x^{(\tau)} {}_u q_{x+t}^{(j)}$

${}_{t|u} q_x^{(j)} = {}_t p_x^{(\tau)} {}_u q_{x+t}^{(j)}$

Multiple decrement table:

$l_{x+t}^{(\tau)} = l_x^{(\tau)} {}_t p_x^{(\tau)} \rightarrow d_{x+t}^{(\tau)} = l_x^{(\tau)} {}_t p_x^{(\tau)} q_{x+t}^{(\tau)}$

$\rightarrow d_{x+t}^{(j)} = l_x^{(\tau)} {}_t p_x^{(\tau)} q_{x+t}^{(j)}$

B2. FRACTIONAL AGE ASSUMPTIONS

CFD between integral ages, for $0 < s + t \leq 1$:

${}_s q_x^{(j)} = \frac{q_x^{(j)}}{q_x^{(\tau)}} \left(1 - \left(p_x^{(\tau)} \right)^s \right)$

${}_s p_{x+t}^{(j)} = \left({}_s p_{x+t}^{(\tau)} \right)^{q_x^{(j)}/q_x^{(\tau)}} \rightarrow q_x^{(j)} = q_x^{(\tau)} \left(\frac{\log p_x'^{(j)}}{\log p_x^{(\tau)}} \right)$

UDD between integral ages in multiple decrement table, for $0 < s + t \leq 1$:

${}_s q_x^{(j)} = s q_x^{(j)}$

${}_s p_{x+t}^{(j)} = \left({}_s p_{x+t}^{(\tau)} \right)^{q_x^{(j)}/q_x^{(\tau)}} \rightarrow q_x^{(j)} = q_x^{(\tau)} \left(\frac{\log p_x'^{(j)}}{\log p_x^{(\tau)}} \right)$

UDD between integral ages in associated single decrement table, for $0 < s + t \leq 1$:

${}_s q_x^{(1)} = s q_x'^{(1)} - \frac{s^2}{2} q_x'^{(1)} q_x'^{(2)}$ Two decrements

${}_s q_x^{(1)} = s q_x'^{(1)} - \frac{s^2}{2} q_x'^{(1)} \left(q_x'^{(2)} + q_x'^{(3)} \right) + \frac{s^3}{3} q_x'^{(1)} q_x'^{(2)} q_x'^{(3)}$ Three decrements

B3. ESTIMATION OF DECREMENT PROBABILITIES

- Define the following:** $T^{(0)}$ is the total waiting time in State 0 between ages x and $x + 1$
- D^{0j} is the number of direct transitions from State 0 to State j between ages x and $x + 1$
- MLE of $\mu_x^{(j)}$:** $\hat{\mu}_x^{(j)} = \frac{D^{0j}}{T^{(0)}}$
- Variance:** $\text{Var}(\hat{\mu}_x^{(j)}) \approx \frac{D^{0j}}{(T^{(0)})^2}$
- Assumption:** $\mu_x^{(j)}$ is constant between ages x and $x + 1$
- MLE of $q_x^{(j)}$:** $\hat{q}_x^{(j)} = 1 - e^{-\hat{\mu}_x^{(j)}}$
- Variance:** $\text{Var}(\hat{q}_x^{(j)}) \approx (1 - \hat{q}_x^{(j)})^2 \frac{D^{0j}}{(T^{(0)})^2}$

B4. TRANSITIONS AT EXACT AGES

For a double decrement model:

Decrement 1 occurs uniformly throughout the year	Decrement 2 occurs only at the beginning of the year	$q_x^{(1)} = p_x^{(2)} q_x^{(1)}$ $q_x^{(2)} = q_x^{(2)}$
	Decrement 2 occurs only at the end of the year	$q_x^{(1)} = q_x^{(1)}$ $q_x^{(2)} = p_x^{(1)} q_x^{(2)}$
	Decrement 2 occurs only at the middle of the year	$q_x^{(1)} = 0.5q_x^{(1)} + 0.5p_x^{(\tau)} 0.5q_{x+0.5}^{(1)} = 0.5q_x^{(1)} + 0.5p_x^{(1)} 0.5p_x^{(2)} 0.5q_{x+0.5}^{(1)}$ $q_x^{(2)} = 0.5q_x^{(2)} = 0.5p_x^{(1)} 0.5q_x^{(2)}$ Note that $0.5q_{x+0.5}^{(2)} = 0$.

B5. INSURANCE AND ANNUITIES, PREMIUMS & RESERVES

- Insurance functions:** $A_x^{(j)} = \sum_{k=0}^{\infty} v^{k+1} {}_k p_x^{(\tau)} q_{x+k}^{(j)}$ $\bar{A}_x^{(j)} = \int_0^{\infty} v^t {}_t p_x^{(\tau)} \mu_{x+t}^{(j)} dt$
- Annuity functions:** $\ddot{a}_x^{(\tau)} = \sum_{k=0}^{\infty} v^k {}_k p_x^{(\tau)}$ $\bar{a}_x^{(\tau)} = \int_0^{\infty} v^t {}_t p_x^{(\tau)} dt$
- Recursive formula:** $({}_k V^g + G - e)(1 + i) = \sum_j (b^{(j)} + E^{(j)}) q_{x+k}^{(j)} + {}_{k+1} V^g p_{x+k}^{(\tau)}$
- $({}_k V^g + G - e)(1 + i)^s = \sum_j (b^{(j)} + E^{(j)}) {}_s q_{x+k}^{(j)} + {}_{k+s} V^g {}_s p_{x+k}^{(\tau)}$ where $0 < s < 1$.

B6. NOTATION

Multiple decrement model	Multiple state model	Statistics
${}_t p_x^{(\tau)}$	${}_t p_x^{00}$	$\Pr(T_x > t)$

${}_tq_x^{(\tau)}$	$\sum_j {}_t p_x^{0j}$	$\Pr(T_x \leq t)$
${}_tq_x^{(j)}$	${}_t p_x^{0j}$	$\Pr(T_x \leq t, J = j)$
${}_tq_x'^{(j)}$	N/A	$\Pr(T_x^{(j)} \leq t)$

Part C: MULTIPLE LIFE MODELS

C1. MULTIPLE LIFE PROBABILITIES

Joint life probabilities: ${}_t p_{xy} = \Pr(T_{xy} > t) = \Pr(T_x > t, T_y > t) = \Pr(\text{Both lives survive } t \text{ years}) = e^{-\int_0^t \mu_{x+s:y+s} dt}$

$${}_t q_{xy} = \Pr(T_{xy} \leq t) = \Pr(\text{At least one life cannot survive } t \text{ years})$$

Last survivor probabilities: ${}_t q_{\overline{xy}} = \Pr(T_{\overline{xy}} \leq t) = \Pr(T_x \leq t) \Pr(T_y \leq t) = \Pr(\text{Both lives die within } t \text{ years})$

$${}_t p_{\overline{xy}} = \Pr(T_{\overline{xy}} > t) = \Pr(\text{At least one life survive } t \text{ years})$$

Formulas:

$${}_{t+u} p_{xy} = {}_t p_{xy} {}_u p_{x+t:y+t}$$

$${}_t |u q_{xy} = {}_t p_{xy} - {}_{t+u} p_{xy} = {}_t p_{xy} {}_u q_{x+t:y+t}$$

$${}_{t+u} p_{\overline{xy}} \neq {}_t p_{\overline{xy}} {}_u p_{x+t:y+t}$$

$${}_t |u q_{\overline{xy}} = {}_t p_{\overline{xy}} - {}_{t+u} p_{\overline{xy}} \neq {}_t p_{\overline{xy}} {}_u q_{x+t:y+t}$$

Contingent probabilities:

$${}_t q_{xy}^1 = \Pr(T_x < T_y, T_x \leq t) = \Pr((x) \text{ dies first and within } t \text{ years})$$

$${}_t q_{xy}^{\cdot 1} = \Pr(T_y < T_x, T_y \leq t) = \Pr((y) \text{ dies first and within } t \text{ years})$$

$${}_t q_{xy}^2 = \Pr(T_y < T_x \leq t) = \Pr((x) \text{ dies second and within } t \text{ years})$$

$${}_t q_{xy}^{\cdot 2} = \Pr(T_x < T_y \leq t) = \Pr((y) \text{ dies second and within } t \text{ years})$$

If the future lifetimes of (x) and (y) are independent:

$$\mu_{x+t:y+t} = \mu_{x+t} + \mu_{y+t}$$

$${}_t p_{xy} = {}_t p_x {}_t p_y$$

$${}_t q_{xy}^1 = \int_0^t {}_s p_x {}_s p_y \mu_{x+t} ds$$

$${}_t q_{\overline{xy}} = {}_t q_x {}_t q_y$$

$${}_t q_{xy}^2 = \int_0^t {}_s p_x {}_s q_y \mu_{x+t} ds$$

Relations:

$${}_t p_{xy} + {}_t q_{xy} = 1$$

$${}_t q_{xy} = {}_t q_{xy}^1 + {}_t q_{xy}^{\cdot 1}$$

$${}_t q_{\overline{xy}} + {}_t p_{\overline{xy}} = 1$$

$${}_t q_{\overline{xy}} = {}_t q_{xy}^2 + {}_t q_{xy}^{\cdot 2}$$

$${}_t p_x + {}_t p_y = {}_t p_{xy} + {}_t p_{\overline{xy}}$$

$${}_t q_x = {}_t q_{xy}^1 + {}_t q_{xy}^2$$

$${}_t q_x + {}_t q_y = {}_t q_{xy} + {}_t q_{\overline{xy}}$$

$${}_t q_y = {}_t q_{xy}^{\cdot 1} + {}_t q_{xy}^{\cdot 2}$$

C2. INSURANCE, ANNUITIES, PREMIUMS & RESERVES

Insurance functions:

\bar{A}_{xy} or A_{xy} = EPV of 1 per year until the first death

$\bar{A}_{\overline{xy}}$ or $A_{\overline{xy}}$ = EPV of 1 per year until the second death

\bar{A}_{1xy} or A_{1xy} = EPV of 1 paid upon (x)'s death if (x) dies first

\bar{A}_{2xy} or A_{2xy} = EPV of 1 paid upon (x)'s death if (x) dies second

Relations:

$$\bar{A}_x + \bar{A}_y = \bar{A}_{xy} + \bar{A}_{\overline{xy}} \qquad A_x + A_y = A_{xy} + A_{\overline{xy}}$$

$$\bar{A}_{xy} = \bar{A}_{1xy} + \bar{A}_{2xy} \qquad A_{xy} = A_{1xy} + A_{2xy}$$

$$\bar{A}_{\overline{xy}} = \bar{A}_{2xy} + \bar{A}_{1xy} \qquad A_{\overline{xy}} = A_{2xy} + A_{1xy}$$

$$\bar{A}_x = \bar{A}_{1xy} + \bar{A}_{2xy} \qquad A_x = A_{1xy} + A_{2xy}$$

Annuity functions:

\bar{a}_{xy} or \ddot{a}_{xy} = EPV of 1 per year until the first death

$\bar{a}_{\overline{xy}}$ or $\ddot{a}_{\overline{xy}}$ = EPV of 1 per year until the second death

$\bar{a}_{x|y}$ or $\ddot{a}_{x|y}$ = EPV of 1 per year until (y) dies – payments begin only after (x) dies

Relations:

$$\bar{a}_x + \bar{a}_y = \bar{a}_{xy} + \bar{a}_{\overline{xy}} \qquad \ddot{a}_x + \ddot{a}_y = \ddot{a}_{xy} + \ddot{a}_{\overline{xy}}$$

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} \qquad \ddot{a}_{x|y} = \ddot{a}_y - \ddot{a}_{xy}$$

Insurance to annuities:

$$\bar{A}_{xy} = \frac{1 - \bar{a}_{xy}}{\delta} \qquad A_{xy} = \frac{1 - \ddot{a}_{xy}}{d}$$

$$\bar{A}_{\overline{xy}} = \frac{1 - \bar{a}_{\overline{xy}}}{\delta} \qquad A_{\overline{xy}} = \frac{1 - \ddot{a}_{\overline{xy}}}{d}$$

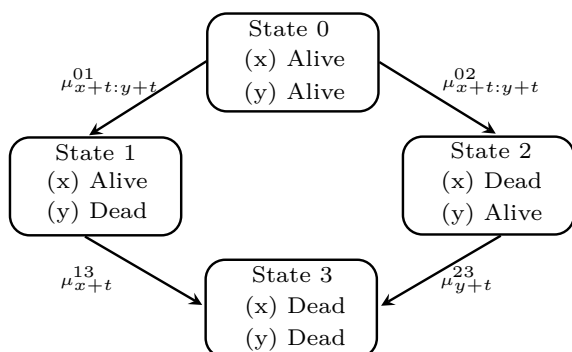
Covariance:

$$Cov(v^{T_{xy}}, v^{T_{\overline{xy}}}) = (\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy})$$

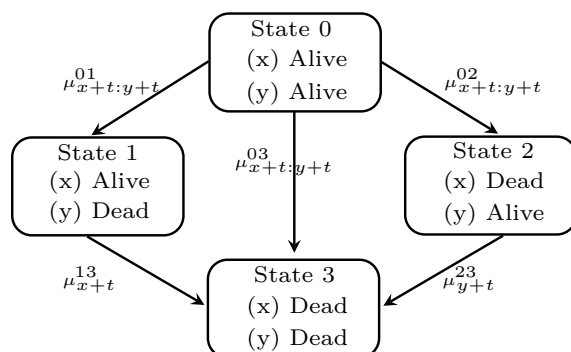
$$Cov(\bar{a}_{T_{xy}}, \bar{a}_{T_{\overline{xy}}}) = (\bar{a}_x - \bar{a}_{xy})(\bar{a}_y - \bar{a}_{xy})$$

C3. MULTIPLE LIFE MODELS

The “classical” multiple life model:



The “classical” common shock model:



Notation and formulas, assuming the “classical” multiple life model:

Multiple life model	Multiple state model	Statistics
${}_t p_{xy}$	${}_t p_x^{00}$	$\Pr(T_x > t, T_y > t)$
${}_t q_{xy}$	${}_t p_x^{01} + {}_t p_x^{02} + {}_t p_x^{03}$	$\Pr(T_x > t, T_y \leq t) + \Pr(T_x \leq t, T_y > t) + \Pr(T_x \leq t, T_y \leq t)$
${}_t q_{\overline{xy}}$	${}_t p_x^{03}$	$\Pr(T_x \leq t, T_y \leq t)$
${}_t p_{\overline{xy}}$	${}_t p_x^{00} + {}_t p_x^{01} + {}_t p_x^{02}$	$\Pr(T_x > t, T_y \leq t) + \Pr(T_x \leq t, T_y > t) + \Pr(T_x > t, T_y > t)$
${}_t p_x$	${}_t p_{xy}^{00} + {}_t p_{xy}^{01}$	$\Pr(T_x > t)$
${}_t q_x$	${}_t p_{xy}^{02} + {}_t p_{xy}^{03}$	$\Pr(T_x \leq t)$
${}_t p_y$	${}_t p_{xy}^{00} + {}_t p_{xy}^{02}$	$\Pr(T_y > t)$
${}_t q_y$	${}_t p_{xy}^{01} + {}_t p_{xy}^{03}$	$\Pr(T_y \leq t)$
${}_t q_{x_1}$	$\int_0^t s p_{xy}^{00} \mu_{x+s:y+s}^{02} ds \neq {}_t p_{xy}^{02}$	$\Pr(T_x \leq t, T_x < T_y)$
${}_t q_{x_1}$	$\int_0^t s p_{xy}^{00} \mu_{x+s:y+s}^{01} ds \neq {}_t p_{xy}^{01}$	$\Pr(T_y \leq t, T_y < T_x)$
${}_t q_{x_2}$	$\int_0^t s p_{xy}^{00} \mu_{x+s:y+s}^{01} {}_{t-s} p_{x+s}^{13} ds \neq {}_t p_{xy}^{03}$	$\Pr(T_x \leq t, T_x > T_y)$
${}_t q_{x_2}$	$\int_0^t s p_{xy}^{00} \mu_{x+s:y+s}^{02} {}_{t-s} p_{x+s}^{23} ds \neq {}_t p_{xy}^{03}$	$\Pr(T_y \leq t, T_y > T_x)$

Part D: PROFIT ANALYSIS

D1. PROFIT TESTING

Recall: Net premiums are always calculated using the equivalence principle.

Equivalence principle → Net premiums are expected to cover benefits.

But gross premiums may not be calculated using the equivalence principle.

Equivalence principle → Gross premiums are expected to cover benefits and expenses.

Otherwise → Gross premiums are expected to cover benefits, expenses and profits (or losses).

Reserve assumptions: Assumptions imposed by government to determine reserves.

Pricing assumptions: Assumptions the company uses to determine premiums.

Profit test assumptions: Assumptions the company uses to conduct profit testing.

For traditional policies, assumptions are mortality (or decrements/transitions), interest rate, expenses, and lapse.

Premiums and reserves are pre-determined or “fixed”.

Define the following:

${}_tV$ is the reserve at time t

G_t is the premium collected at time t

E_t is the annual expenses incurred at time t

i_t is the interest rate earned on the investments from time t to time $t + 1$

DB_t is the death benefit payable at time t

E_t^{DB} is the expenses associated with death benefit incurred at time t

Expected profit:

$$\text{Pr}_{t+1} = ({}_tV + G_t - E_t)(1 + i_t) - (DB_{t+t} + E_{t+1}^{DB})q_{x+k} - {}_{t+1}Vp_{x+t}$$

This formula is for alive-dead model and can be modified for MS/MD/ML models.

D2. PROFIT MEASURES

Profit vector:

Pr_k

Pr_0 usually contains pre-contract expenses only.

Profit signature:

$$\Pi_k = {}_{k-1}p_x \text{Pr}_k$$

Note that $\Pi_0 = \text{Pr}_0$.

Internal rate of return:

IRR

Such that $\sum_{k=0}^n \frac{\Pi_k}{(1 + IRR)^k} = 0$.

Net present value:

$$NPV = \sum_{k=0}^n \frac{\Pi_k}{(1 + r)^k}$$

Where r is the hurdle rate.

Partial NPV:

$$NPV(t) = \sum_{k=0}^t \frac{\Pi_k}{(1 + r)^k}$$

Where r is the hurdle rate.

Discounted payback period:

Smallest m

Such that $NPV(m) = \sum_{k=0}^m \frac{\Pi_k}{(1 + r)^k} \geq 0$.

Profit margin:

$$M = \frac{NPV}{EPV(\text{Premiums})}$$

Where $EPV(\text{Premiums})$ is calculated using r , the hurdle rate.

Premiums can be determined using a desired profit margin.

Reserves can be determined using a process called **zeroization**.

D3. ACTUAL PROFIT & GAIN BY SOURCE

Expected profit: Profit(Expected interest, Expected expense, Expected mortality)

Actual profit: Profit(Actual interest, Actual expense, Actual mortality)

Overall gain: Overall gain = Actual profit - Expected profit

The usual order of **gain by source**: Interest → Expense → Mortality

$$\text{Gain from interest} = \text{Profit(A.interest, E.expense, E.mortality)} - \text{Profit(E.interest, E.expense, E.mortality)}$$

$$\text{Gain from expense} = \text{Profit(A.interest, A.expense, E.mortality)} - \text{Profit(A.interest, E.expense, E.mortality)}$$

$$\text{Gain from mortality} = \text{Profit(A.interest, A.expense, A.mortality)} - \text{Profit(A.interest, A.expense, E.mortality)}$$

$$\text{Overall gain} = \text{Gain due to interest} + \text{Gain due to expense} + \text{Gain due to mortality}$$

Part E: UNIVERSAL LIFE INSURANCE

E1. UL ACCOUNT VALUES

Define the following:

AV_t is the account value at time t

P_t is the premium collected at time t

EC_t is the expense charge deducted from the account value at time t

i_t^c is the credited interest rate earned on the investments time t to time $t + 1$

CoI_t is the cost of insurance charge at time t

q_{x+t}^* is the CoI mortality rate

i^* is the CoI interest rate

FA is the face amount

SC_t is the surrender charge deducted from account value at time t

	Type A UL	Type B UL
Account Value	$(AV_t + P_t - EC_t)(1 + i_t^c)$ $= \frac{1 + i_t^c}{1 + i^*} q_{x+t}^* FA + \left(1 - \frac{1 + i_t^c}{1 + i^*} q_{x+t}^*\right) AV_{t+1}$	$(AV_t + P_t - EC_t)(1 + i_t^c)$ $= \frac{1 + i_t^c}{1 + i^*} q_{x+t}^* FA + AV_{t+1}$
Cost of Insurance	$CoI_{t+1} = \frac{1}{1 + i^*} q_{x+t}^* (FA - AV_{t+1})$	$CoI_{t+1} = \frac{1}{1 + i^*} q_{x+t}^* FA$
Additional Death Benefit	$ADB_{t+1} = FA - AV_{t+1}$	$ADB_{t+1} = FA$
Death Benefit	$DB_{t+1} = FA$	$DB_{t+1} = FA + AV_{t+1}$
Surrender Benefit	$SB_t = AV_{t+1} - SC_{t+1}$	$SB_t = AV_{t+1} - SC_{t+1}$

E2. UL CORRIDOR REQUIREMENT

Corridor factor: γ

Type A UL	Type B UL
<ol style="list-style-type: none"> Calculate AV_{t+1} using the above formula. If $\gamma AV_{t+1} \leq FA$, good. If $\gamma AV_{t+1} > FA$, replace FA with γAV_{t+1} in the formula and calculate AV_{t+1} again. The revised death benefit is γAV_{t+1}. 	<ol style="list-style-type: none"> Calculate AV_{t+1} using the above formula. If $\gamma AV_{t+1} \leq FA + AV_{t+1}$, good. If $\gamma AV_{t+1} > FA + AV_{t+1}$, replace $FA + AV_{t+1}$ with γAV_{t+1} in the formula and calculate AV_{t+1} again. The revised death benefit is γAV_{t+1}.

E3. UL RESERVES

EPV of Type A UL death benefit:

$$(FA)A_{x+t:n-t}$$

Type A UL no lapse guarantee reserve:

$${}_tV^{NLG} = \max\left(0, (FA)A_{x+t:n-t} - AV_t\right)$$

E4. UL PROFIT TESTING

Define the following: E_t is the annual expenses incurred at time t

E_t^{DB} is the expenses associated with death benefit incurred at time t

E_t^{SB} is the expenses associated with surrender benefit incurred at time t

i_t is the interest rate earned on the investment from time t to time $t + 1$

Expected profit: $Pr_{t+1} = (AV_{t-1} + P_t - E_t)(1 + i_t) - (DB_{t+1} + E_{t+1}^{DB})q_{x+t}^{(d)} - (SB_{t+1} + E_{t+1}^{SB})q_{x+t}^{(w)} - AV_{t+1}p_{x+t}^{(\tau)}$

Part F: EMBEDDED OPTIONS AND EQUITY-LINKED INSURANCE

F1. EMBEDDED OPTIONS

Four popular guarantees:

Guaranteed minimum death benefit (GMDB)

Guaranteed minimum accumulation/maturity benefit (GMAB/GMMB)

Guaranteed minimum withdrawal benefit (GMWB)

Guaranteed minimum income benefit (GMIB)

	Payoff and value of guarantee	Where
GMDB with a return of premium guarantee	Time- T payoff = $\max(0, K - S_T)$ Time-0 value = $\int_0^\infty P(K, t) f_T(t) dt$	T is the future lifetime of policyholder S_T is the AV at time T K is the amount invested $P(K, t)$ is the price of a K -strike put option expired at time t
Earnings-enhanced death benefit	Time- T payoff = $\alpha \max(0, S_T - K)$ Time-0 value = $\alpha \int_0^\infty C(K, t) f_T(t) dt$	T is the future lifetime of policyholder α is the guarantee percentage S_T is the AV at time T K is the amount invested $C(K, t)$ is the price of a K -strike call option expired at time t
GMAB/GMMB with a return of premium guarantee	Time- τ payoff = $\max(0, K - S_\tau)$ for $T^* > \tau$ Time-0 value = $P(K, \tau) \times \Pr(T^* > \tau)$	T^* is the future lifetime of policy τ is the guarantee period S_τ is the AV at time τ K is the amount invested $P(K, \tau)$ is the price of a K -strike put option expired at time τ

Define the following:

F_t is the policyholder's fund at time t

n is the maturity of the contract

P is the single premium

e is the expense charge

m is the management charge

S_t is the stock index, with $S_0 = 1$

$$\xi = (1 - e)(1 - m)^{n-1}$$

$h(t)$ is the option payoff at time t

$$\nu(0, t) = E_0^Q [e^{-rt} h(t)]$$

GMMB

Time 0 value:

$$\pi(0) = P_n p_x (e^{-rn} \Phi(-d_2) - \xi \Phi(-d_1))$$

$$d_1 = \frac{\ln \xi + (r + 0.5\sigma^2)n}{\sigma\sqrt{n}}$$

$$d_2 = d_1 - \sigma\sqrt{n}$$

Time t value (reserve) :

$$\pi(t) = P_{n-t} p_{x+t} (e^{-r(n-t)} \Phi(-d_2) - S_t \xi \Phi(-d_1))$$

$$d_1 = \frac{\ln S_t \xi + (r + 0.5\sigma^2)(n-t)}{\sigma \sqrt{n-t}}$$

$$d_2 = d_1 - \sigma \sqrt{n-t}$$

GMDB

Time 0 value (continuous):

$$\pi(0) = \int_0^n v(0, t) {}_t p_x \mu_{x+t} dt$$

Time 0 value (monthly):

$$\pi(0) = \sum_{t=0}^{12n-1} v(0, t/12) {}_{\frac{t-1}{12}} p_x \frac{1}{12} q_{x+\frac{t-1}{12}}$$

F2. HEDGING AND REBALANCING

GMMB

Hedge Portfolio

Time t value

$$\pi(t) = {}_{n-t} p_{x+t} E_0^Q [e^{-r(n-t)} h(n)] = {}_{n-t} p_{x+t} \nu(t, n)$$

Stock part

$${}_{n-t} p_{x+t} \left(\frac{d}{dS_t} \nu(t, n) \right) S_t$$

Zero-coupon bond part

$$\pi(t) - {}_{n-t} p_{x+t} \left(\frac{d}{dS_t} \nu(t, n) \right) S_t$$

GMDB

Hedge Portfolio

Time t value

$$\pi(t) = \int_0^{n-t} \nu(t, w+t) {}_w p_{x+t} \mu_{x+t+w} dw$$

Stock part

$$\int_0^{n-t} S_t \left(\frac{d}{dS_t} \nu(t, w+t) \right) {}_w p_{x+t} \mu_{x+t+w} dw$$

Zero-coupon bond part

$$\int_0^{n-t} \left(\nu(t, w+t) - S_t \left(\frac{d}{dS_t} \nu(t, w+t) \right) \right) {}_w p_{x+t} \mu_{x+t+w} dw$$

F3. PROFIT TESTING

Columns of profit test if company purchases options:

- + Initial expense charge in the first period, if any
- + Management charge
- Expenses
- Risk premium
- + Release of beginning reserve, if any
- + Interest, if any
- Reserve per survivor at end of period, if any

Columns of profit test if company hedges internally:

- + Initial expense charge in the first period, if any
- + Management charge
- Expenses
- + Release of beginning reserve, if any
- + Interest, if any
- GMxB cost
- Hedging/rehedging cost
- Reserve per survivor at end of period, if any

F4. EQUITY-LINKED INSURANCE

Define the following:

- F_t is the policyholder's fund at time t
- P_t is the premium paid at time t
- AP_t is allocated premium - the part of P_t that is invested in the fund
- i_t^f is the interest rate earned on the fund from time t to time $t + 1$
- MC_t is the management charge deducted from the fund at time t

General Formula:

$$F_{t+1} = (F_t + AP_t) (1 + i_t^f) - MC_{t+1}$$

Define the following:

- ${}_tV$ is the reserve at time t
- $UAP_t = P_t - AP_t$ is the unallocated premium
- E_t is the expenses incurred at time t
- i_t^a is the interest rate earned on the insurer's assets from time t to time $t + 1$
- DB_t is the death benefit payable at time t
- CV_t is the cash value at time t

Expected profit:

$$\begin{aligned} Pr_{t+1} = & ({}_tV + UAP_t - E_t) (1 + i_t^a) + MC_{t+1} - (DB_{t+1} - F_{t+1}) q_{x+t}^{(d)} \\ & - (CV_{t+1} - F_{t+1}) q_{x+t}^{(w)} - {}_{t+1}V p_{x+t}^{00} \end{aligned}$$

Part G: PENSIONS

G1. SERVICE TABLE

ALTAM service table:	Death d_x	
	Disability i_x	
	Withdrawal w_x	
	Retirement r_x	The service table is a multiple decrement table.

G2. SALARY SCALE & REPLACEMENT RATIO

Salary rate:	\bar{S}_x	Salary rate at exact age x .
Salary scale:	$s_x = \int_0^1 \bar{s}_{x+t} dt$	Salary rate during the year $[x, x + 1)$.
Approximation:	$\bar{S}_x \approx S_{x-0.5}$	
Salary:	$S_y = S_x \frac{s_y}{s_x}$	Salary received during the year $[y, y + 1)$.
Replacement ratio:	$R = \frac{\text{Pension income in the year after retirement}}{\text{Salary in the year before retirement}}$	
Define the following:	xe is the age the member enters the plan	
	xr is the age the member retires	
	$n_{xr} = xr - xe$ is the total number of years of service	
	$TPE_{xr} = S_{xe} + S_{xe+1} + \dots + S_{xr-1}$ is the total pensionable earnings .	
Final average salary:	$S_{xr}^F = \frac{S_{xr-k} + \dots + S_{xr-2} + S_{xr-1}}{k}$	The average of salaries during the final k years.
Career average salary:	$S_{xr}^C = \frac{TPE_{xr}}{n_{xr}}$	The average of salaries during the whole career.

G3. DEFINED BENEFIT PLANS

Accrual rate of DB plan:	α	
DB plan annual benefit:	$B = n_{xr} \alpha S_{xr}^F$	Final average salary
	$B = n_{xr} \alpha S_{xr}^C = \alpha TPE_{xr}$	Career average salary
Sponsor of DB plan:	If $B = n_{xr} \alpha S_{xr}^F$ is payable per year right after retirement and until the member dies, then the sponsor has to purchase an annuity worth $B \ddot{a}_{xr}$ for the member. Early retirement benefits may be subject to a reduction factor RF_{xr} .	

APV of DB plan:	$APV = \sum_{xr} n_{xr} \times \alpha \times S_{xr}^F \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$	Final average salary
	$APV = \sum_{xr} \alpha \times TPE_{xr} \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$	Career average salary
AL of DB plan, PUC method:	$AL = \sum_{xr} n_x \times \alpha \times S_{xr}^F \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$	Final average salary
	$AL = \sum_{xr} n_x \times \alpha \times \frac{TPE_{xr}}{n_{xr}} \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$	Career average salary
AL of DB plan, TUC method:	$AL = \sum_{xr} n_x \times \alpha \times S_x^F \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$	Final average salary
	$AL = \sum_{xr} \alpha \times TPE_x \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$	Career average salary
Normal contribution:	$C = v p_x^{00} {}_1V - {}_0V + EPV(\text{Benefits for mid-year exits})$	
No exit benefits, PUC method:	$C = {}_0V \left(\frac{1}{n}\right)$	No exit benefits, and ${}_tV$ calculated using PUC method.
No exit benefits, TUC method:	$C = {}_0V \left(\frac{S_{x+1} n+1}{S_x n} - 1\right)$	No exit benefits, and ${}_tV$ calculated using TUC method.

G4. RETIREE HEALTHCARE PLANS

Premium for healthcare plan:	$B(x, t)$	Premium paid at time t for a person aged x
Benefit premium annuity:	$\ddot{a}_B(x, t) = \sum_{k=0}^{\infty} \frac{B(x+k, t+k)}{B(x, t)} v^k {}_k p_x$	This is an increasing annuity-due.
Simplification:	$\ddot{a}_B(x, t) = \sum_{k=0}^{\infty} c^k (1+j)^k v^k {}_k p_x$	Where: $c = \frac{B(x+1, t)}{B(x, t)}$
	$\ddot{a}_B(x, t) = \ddot{a}_{x@i^*}$	$1+j = \frac{B(x, t+1)}{B(x, t)}$
		$1+i^* = \frac{1+i}{c(1+j)}$

EPV of retiree healthcare plan: $EPV = B(x, t)\ddot{a}_B(x, t)$

For someone retires at age x and time t .

AVTHB:

$$AVTHB = \sum_{t=0}^{65-x} B(x+t, t)\ddot{a}_B(x+t, t) \times v^t \times \frac{r_{x+t}}{l_x}$$

$$AVTHB = B(x, 0) \sum_{t=0}^{65-x} \ddot{a}_{x+t@i^*} \times v^{*t} \times \frac{r_{x+t}}{l_x}$$

Assume retirement must occur no later than age 65.

APBO:

$${}_0V = \sum_{t=0}^{65-x} B(x+t, t)\ddot{a}_B(x+t, t) \times v^t \times \frac{r_{x+t}}{l_x} \times \frac{x-x_0}{x+t-x_0}$$

Where x_0 is the age the employer begins working.

Normal cost:

$$C = v p_x {}_1V - {}_0V + EPV(\text{Benefits for mid-year exits})$$

No exit benefits.

$$C = {}_0V\left(\frac{1}{n}\right)$$

Where n is the number of years of service